FROM ATIYAH CLASSES TO HOMOTOPY LEIBNIZ ALGEBRAS

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Abstract. A celebrated theorem of Kapranov states that the Atiyah class of the tangent bundle of a complex manifold X makes $T_X[-1]$ into a Lie algebra object in the derived category $D^+(X)$. Furthermore, for X Kähler, Kapranov proved that the Dolbeault resolution $\Omega^{\bullet-1}(T_X^{1,0})$ of $T_X[-1]$ is an L_∞ algebra. In this paper, we prove that Kapranov's theorem holds in much wider generality for vector bundles over Lie pairs. Given a Lie pair (L, A), i.e. a Lie algebroid L together with a Lie subalgebroid A, we define the Atiyah class α_E of an A-module E as the obstruction to the existence of an A-*compatible* L-connection on E. We prove that the Atiyah classes $\alpha_{L/A}$ and α_E respectively make $L/A[-1]$ and $E[-1]$ into a Lie algebra and a Lie algebra module in the derived category $D^+(\mathcal{A})$ of the category $\mathcal{A}$ of left $\mathcal{U}(A)$-modules, where $\mathcal{U}(A)$ is the universal enveloping algebra of A. Moreover, we produce a homotopy Leibniz algebra and a homotopy Leibniz module stemming from the Atiyah classes of L/A and E, and inducing the aforesaid Lie structures in $D^+(\mathcal{A})$.

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Introduction

The Atiyah class of a holomorphic vector bundle E over a complex manifold X, as initially introduced by Atiyah [3], constitutes the obstruction to the existence of a holomorphic connection on said holomorphic vector bundle. It is constructed in the following way. The vector bundle $\mathscr{J}^1 E$ of jets (of order 1) of holomorphic sections of $E \to X$ fits into the canonical short exact sequence $0 \to T_X^* \otimes E \to \mathscr{J}^1 E \to E \to 0$ of holomorphic vector bundles (over the complex manifold X). The Atiyah class of $E \to X$ is the extension class $\alpha_E \in \operatorname{Ext}^1_X(E, T_X^* \otimes E) \cong H^1(X; T_X^* \otimes \operatorname{End} E)$ of this short exact sequence [3, 22].

In the late 1990's, Rozansky and Witten proposed a construction of a family of new 3-dimensional topological quantum field theories, indexed by compact (or asymptotically flat) hyper-Kähler manifolds [40]. The Rozansky-Witten procedure does thus associate topological invariants of 3-manifolds to each hyper-Kähler manifold. In subsequent work, Kapranov [22] and Kontsevich [24] revealed the crucial role played by Atiyah classes in the construction of the Rozansky-Witten invariants. They [22, 24] in particular showed that the hyper-Kähler restriction is unnecessary and that the theory devised by Rozansky and Witten works for all holomorphic symplectic manifolds. In Kapranov's work lies the key fact that the Atiyah class of the tangent bundle of a complex manifold X yields a map $T_X[-1] \otimes T_X[-1] \to T_X[-1]$ in the derived category $D^+(X)$ of bounded below complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology, which makes $T_X[-1]$ into a Lie algebra object in $D^+(X)$. Therefore, Kapranov's approach shone light on many similarities between the Rozansky-Witten and Chern-Simons theories [4, 5] as shown by Roberts and Willerton [39].

Atiyah classes have also enjoyed renewed vigor due to Kontsevich's seminal work on deformation quantization [25, 23]. Kontsevich indicated the existence of deep ties between the Todd genus of complex manifolds and the Duflo element in Lie theory [25, 23, 41, 11]. This discovery inspired several subsequent works on Hochschild (co)homology and the Hochschild-Kostant-Rosenberg isomorphism for complex manifolds, by Dolgushev, Tarmarkin and Tsygan [17, 16], Căldăraru [15], Markarian [33], Ramadoss [37], and Calaque and Van den Bergh [12] among many others. In particular, the work of Markarian [33] (see also Ramadoss [37]) led to an alternative proof of the Hirzebruch-Riemann-Roch theorem and its variations.

In [35, 36], Molino introduced an Atiyah class for connections "transverse to a foliation", which measures the obstruction to their "projectability". Molino class has many applications in geometry, for instance, in the study of differential operators on a foliate manifold [44], and of deformation quantization theory [6].

This paper is the first in a sequence of works which aims at developing a theory of Atiyah classes in a general setting and studying their applications. Our goal is to explore emerging connections between derived geometry and classical areas of mathematics such as complex geometry, foliation theory, Poisson geometry and Lie theory. The present paper develops a framework which encompasses both the original Atiyah class of holomorphic vector bundles and the Molino class of real vector bundles foliated over a foliation as special cases.

Lie algebroids are the starting point of our approach. Indeed, holomorphic vector bundles and vector bundles foliated over a foliation may both be seen as instances of the concept of module over a Lie algebroid, a straightforward generalization of the well-known representations of Lie algebras. Given a Lie algebroid L over a base manifold M, an L-connection on a vector bundle $E \to M$ is a bilinear map $X \otimes s \mapsto \nabla_X s$ from $\Gamma(L) \otimes \Gamma(E)$ to $\Gamma(E)$ satisfying the usual axioms $\nabla_{fX} s = f \nabla_X s$ and $\nabla_X(fs) = \rho(X)f \cdot s + f \nabla_X s$ for any $f \in C^\infty(M)$. If the connection is flat, E is said to be a module over the Lie algebroid L. When the base is the one-point space, the L-modules are simply Lie algebra modules in the

classical sense. When the base is a complex manifold X, the holomorphic vector bundles over a complex manifold X are known to be the modules of the complex Lie algebroid $T_X^{0,1}$ stemming from the complex manifold X. Molino's foliated bundles are modules over the Lie algebroid structure carried by the characteristic distribution of the foliation of their base.

We introduce a general theory of Atiyah classes of vector bundles over Lie algebroid pairs. By a Lie algebroid pair (L, A), we mean a Lie algebroid L (over a manifold M) together with a Lie subalgebroid A (over the same base M) of L. And by a vector bundle E over the Lie algebroid pair (L, A), we mean a vector bundle E (over M), which is a module over the Lie subalgebroid A. Given such a Lie algebroid pair (L, A) and A-module E, we consider the jet bundle $\mathscr{J}^1_{L/A}E$ (of order 1), whose smooth sections are the L-connections on E extending the (infinitesimal) A-action on E in a compatible way. We prove the following

Theorem (A)**.** *The jet bundle $\mathscr{J}^1_{L/A}E$ is naturally an A-module. It fits in a short exact sequence of A-modules $0 \to A^\perp \otimes E \to \mathscr{J}^1_{L/A}E \to E \to 0$. Here $A^\perp$ denotes the annihilator of A in L.*

We call the extension class $\alpha_E \in \operatorname{Ext}^1_{\mathcal{A}}(E, A^\perp \otimes E) \cong H^1(A, A^\perp \otimes \operatorname{End} E)$ of this short exact sequence, the Atiyah class of E because, when $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$ for a complex manifold X, $\mathscr{J}^1_{L/A}E$ is the space of 1-jets of holomorphic sections of E and α_E is its (classical) Atiyah class; and, when L is the tangent bundle of a smooth manifold M and A is an integrable distribution on M, α_E is the Molino class of the vector bundle E foliated over A. Geometrically, the Atiyah class can thus be interpreted as the obstruction to the existence of a compatible L-connection on E which extends the A-action with which E is endowed.

It turns out that the Atiyah class introduced in our general context and the classical Atiyah class of holomorphic vector bundles enjoy similar rich algebraic properties.

We denote the (abelian) category of modules over the universal enveloping algebra $\mathcal{U}(A)$ of the Lie algebroid A by the symbol $\mathcal{A}$. Every vector bundle over M endowed with an A-action — more pecisely its space of smooth sections — is an object of $\mathcal{A}$. The bounded below derived category of $\mathcal{A}$ will be denoted by $D^+(\mathcal{A})$.

Given a Lie algebroid pair (L, A), the quotient L/A is naturally an A-module. When L is the tangent bundle to a manifold M and A is an integrable distribution on M, the A-action on L/A is given by the Bott connection [8]. We consider L/A as a complex concentrated in degree 1 and refer to it as $L/A[-1]$. We show that the Atiyah class of L/A makes $L/A[-1]$ into a Lie algebra object in the derived category $D^+(\mathcal{A})$. Indeed, being an element of

$$\operatorname{Ext}^1_{\mathcal{A}}(L/A, A^\perp \otimes L/A) \cong \operatorname{Ext}^1_{\mathcal{A}}(L/A \otimes L/A, L/A) \cong \operatorname{Hom}_{D^+(\mathcal{A})}(L/A[-1] \otimes L/A[-1], L/A[-1]),$$

the Atiyah class $\alpha_{L/A}$ of the A-module L/A defines a "Lie bracket" on $L/A[-1]$. If, moreover, E is an A-module, its Atiyah class

$$\alpha_E \in \operatorname{Ext}^1_{\mathcal{A}}(E, A^\perp \otimes E) \cong \operatorname{Ext}^1_{\mathcal{A}}(L/A \otimes E, E) \cong \operatorname{Hom}_{D^+(\mathcal{A})}(L/A[-1] \otimes E[-1], E[-1])$$

defines a "representation" on $E[-1]$ of the "Lie algebra" $L/A[-1]$. In summary, we prove the following

Theorem (B)**.** *Let (L, A) be a Lie algebroid pair with quotient L/A. Then the Atiyah class of L/A makes $L/A[-1]$ into a Lie algebra object in the derived category $D^+(\mathcal{A})$. Moreover, if E is an A-module, then $E[-1]$ is a module object over the Lie algebra object $L/A[-1]$ in the derived category $D^+(\mathcal{A})$.*

The above result suggests that, on the cochain level, the Atiyah class should define some kind of Lie algebra up to a certain homotopy. But how does one obtain a cocycle representing the Atiyah class? Recall that a Dolbeault representative of the Atiyah class of a holomorphic vector bundle $E \to X$ can be obtained in the following way. Considering $T_X^{1,0}$ as a complex Lie algebroid, choose a $T_X^{1,0}$-connection $\nabla^{1,0}$ on E. Being a holomorphic vector bundle, E carries a canonical flat $T_X^{0,1}$-connection $\overline{\partial}$. Adding $\nabla^{1,0}$ and $\overline{\partial}$, we obtain a $T_X \otimes \mathbb{C}$-connection ∇ on E. The element $\mathcal{R} \in \Omega^{1,1}(\operatorname{End} E)$ defined by $\mathcal{R}(X^{0,1}; Y^{1,0})s = \nabla_{X^{0,1}}\nabla_{Y^{1,0}}s - \nabla_{Y^{1,0}}\nabla_{X^{0,1}}s - \nabla_{[X^{0,1},Y^{1,0}]}s$ is a Dolbeault 1-cocycle whose cohomology class (which is independent of the choice of $\nabla^{1,0}$) is the Atiyah class $\alpha_E \in H^{1,1}(X, \operatorname{End} E) \cong H^1(X, T_X^* \otimes \operatorname{End} E)$. In the more general setting of vector bundles over Lie algebroid pairs, the Atiyah class can be defined in terms of Lie algebroid connections as follows. Assume (L, A) is a Lie pair, and E is an A-module. Let ∇ be any L-connection on E extending its A-action. The curvature of ∇ is the bundle map $R^\nabla : \wedge^2 L \to \operatorname{End} E$ defined by $R^\nabla(l_1, l_2) = \nabla_{l_1}\nabla_{l_2} - \nabla_{l_2}\nabla_{l_1} - \nabla_{[l_1,l_2]}$, for all $l_1, l_2 \in \Gamma(L)$. Since E is an A-module, its restriction to $\wedge^2 A$ vanishes. Hence the curvature induces a section $R_E^\nabla \in \Gamma(A^* \otimes A^\perp \otimes \operatorname{End} E)$, which is a 1-cocycle for the Lie algebroid A with values in the A-module $A^\perp \otimes \operatorname{End} E$. We prove that the cohomology class $\alpha_E \in H^1(A; A^\perp \otimes \operatorname{End} E)$ of the 1-cocycle R_E^∇ is precisely the Atiyah class of the A-module E.

When the A-module E is the quotient L/A of the Lie algebroid pair (L, A), by choosing an L-connection ∇ on L/A extending the A-action, we get the Atiyah cocycle $R_{L/A}^\nabla \in \Gamma(A^* \otimes (L/A)^* \otimes \operatorname{End}(L/A))$, which may be regarded as a bundle map $R_2 : L/A \otimes L/A \to A^* \otimes L/A$. Starting from R_2 and a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$, a sequence $(R_n)_{n=2}^\infty$ of bundle maps

$$R_n : \otimes^n L/A \to \operatorname{Hom}(A, L/A)$$

can be defined inductively by the relation $R_{n+1} = \partial^\nabla R_n$, for $n \geq 2$. Alternatively, R_n can be seen as a section of the vector bundle $A^* \otimes (\otimes^n (L/A)^*) \otimes L/A$. Then the graded vector space $\bigoplus_{n=0}^\infty \Gamma(\wedge^n A^* \otimes L/A)$ can be endowed with a sequence $(\lambda_k)_{k=1}^\infty$ of multibrackets $\lambda_k : \otimes^k V \to V$: the unary bracket λ_1 is chosen to be coboundary operator ∂^A of exterior forms on the Lie algebroid A taking values in the A-module L/A, while all higher order brackets λ_k are defined by the relation

$$\lambda_k(\xi_1 \otimes b_1, \cdots, \xi_k \otimes b_k) = (-1)^{|\xi_1|+\cdots+|\xi_k|} \xi_1 \wedge \cdots \wedge \xi_k \wedge R_k(b_1, \cdots, b_k),$$

where $b_1, \ldots, b_k \in \Gamma(L/A)$ and $\xi_1, \ldots, \xi_k$ are arbitrary homogeneous elements of $\Gamma(\wedge^\bullet A^*)$.

By an A-algebra, we mean a bundle of associative algebras $\mathcal{C}$ over M which is an A-module, and on which $\Gamma(A)$ acts by derivations. For a commutative A-algebra $\mathcal{C}$, $(\lambda_k)_{k=1}^\infty$ extends in a natural way to the graded subspace $\bigoplus_{n=0}^\infty \Gamma(\wedge^n A^* \otimes L/A \otimes \mathcal{C})$. We prove

Theorem (C)**.** *Assume that (L, A) is a Lie pair and $\mathcal{C}$ is a commutative A-algebra. When endowed with the sequence of multibrackets $(\lambda_k)_{k\in\mathbb{N}}$, the graded vector space $\Gamma(\wedge^\bullet A^* \otimes L/A \otimes \mathcal{C})[-1]$ becomes a Leibniz$_\infty$ algebra — a natural generalization of Stasheff's L_∞ algebras* [28] *first introduced by Loday* [30] *in which the requirement that the multibrackets be (skew-)symmetric is dropped.*
If E is an A-module, the graded vector space $\Gamma(\wedge^\bullet A^* \otimes E \otimes \mathcal{C})[-1]$ becomes a Leibniz$_\infty$ module over the Leibniz$_\infty$ algebra $\Gamma(\wedge^\bullet A^* \otimes L/A \otimes \mathcal{C})[-1]$.
As a consequence, $\bigoplus_{i\geq 1} H^{i-1}(A, L/A \otimes \mathcal{C})$ is a graded Lie algebra and $\bigoplus_{i\geq 1} H^{i-1}(A, E \otimes \mathcal{C})$ a module over it.

We also identify a simple criterion for detecting when this Leibniz algebra is actually an L_∞ algebra. This situation happens when X is a Kähler manifold, $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$. Then we recover the L_∞-structure on $\Omega^{0,\bullet-1}(T^{1,0})$ discovered by Kapranov [22].

The intrinsic meaning of this homotopic algebraic structure arising from our construction of the Atiyah class, and its relation with the L_∞-space of Costello [13, 14] will be explored somewhere else.

Note that our definition of the Atiyah class could easily be generalized to complexes of A-modules as in [33]. We also refer the interested reader to [12] for the Atiyah class of a DG-module over a dDG-algebra. After the first draft of this paper was posted on arXiv, Calaque inferred that, for Lie algebra pairs $(\mathfrak{d}, \mathfrak{g})$, i.e. Lie algebroid pairs with the one-point space as base manifold, the Atiyah class of the quotient $\mathfrak{d}/\mathfrak{g}$ coincides with the class capturing the obstruction to the "PBW problem" studied earlier by Calaque-Căldăraru-Tu [10] (see also [18]). Bordemann gave a nice interpretation of the Calaque-Căldăraru-Tu class as the obstruction to the existence of invariant connections on homogeneous spaces [7]. Another very recent development is Calaque's beautiful work [9] on the relation between the Atiyah class of the A-module L/A with respect to the Lie pair (L, A) and the relative PBW problem previously solved by Căldăraru-Calaque-Tu [10]. Calaque also pointed out to us that our results should be related to the obstruction to a relative HKR isomorphism for closed embeddings of algebraic varieties identified by Arinkin-Căldăraru [2]. This certainly deserves further investigation. Finally, we would like to mention, in relation to the homotopy algebra results of the present paper, Yu's very interesting doctoral thesis on L_∞-algebroids [45]. Stasheff's work on constrained Poisson algebras [42] is another interesting result which could well be related to the present paper.

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1. Preliminaries: connections, modules, Lie pairs, and matched pairs

Let M be a smooth manifold, let $L \to M$ be a Lie algebroid, and let $E \xrightarrow{\pi} M$ be a vector bundle. The anchor map of L is denoted by ρ.

Recall that the Lie algebroid differential $d : \Gamma(\wedge^\bullet L^*) \to \Gamma(\wedge^{\bullet+1} L^*)$ is given by

$$\big(d\mu\big)(x_0, \cdots, x_n) = \sum_{i=0}^{n} (-1)^i \rho(x_i)\big(\mu(x_0, \cdots, \widehat{x_i}, \cdots, x_n)\big) + \sum_{i<j} (-1)^{i+j} \mu([x_i, x_j], x_0, \cdots, \widehat{x_i}, \cdots, \widehat{x_j}, \cdots, x_n).$$

The traditional description of a (linear) L-connection on E is in terms of a covariant derivative

$$\Gamma(L) \times \Gamma(E) \to \Gamma(E) : (x, e) \mapsto \nabla_x e$$

characterized by the following two properties:

$$\nabla_{fx} e = f \nabla_x e,$$
$$\nabla_x (fe) = \rho(x) f \cdot e + f \cdot \nabla_x e,$$

for all $x \in \Gamma(L)$, $e \in \Gamma(E)$, and $f \in C^\infty(M)$.

Here, we give three equivalent descriptions of (linear) L-connections on E: covariant differential, horizontal lifting, and horizontal distribution.

Definition 1. *A (linear) L-connection on E is a map $\Gamma(E) \xrightarrow{d^\nabla} \Gamma(L^* \otimes E)$, called covariant differential, satisfying the Leibniz rule*

$$d^\nabla(fe) = \rho^*(df) \otimes e + f \cdot d^\nabla e,$$

for all $f \in C^\infty(M)$ and $e \in \Gamma(E)$.

The covariant differential $\Gamma(E) \xrightarrow{d^\nabla} \Gamma(L^* \otimes E)$ extends uniquely to a degree 1 operator

$$\Gamma(\wedge^\bullet L^* \otimes E) \xrightarrow{d^\nabla} \Gamma(\wedge^{\bullet+1} L^* \otimes E)$$

satisfying the Leibniz rule

$$d^\nabla(\beta \otimes e) = d\beta \otimes e + (-1)^b \beta \otimes d^\nabla e,$$

for all $\beta \in \Gamma(\wedge^b L^*)$ and $e \in \Gamma(E)$.

Definition 2. *A (linear) L-connection on E is a map $L \times_M E \xrightarrow{h} T_E$, called horizontal lifting, such that the diagram*

$$\begin{tikzcd}
 & & L \arrow[rr, "\rho"] \arrow[dr] & & T_M \arrow[dl] \\
L \times_M E \arrow[urr] \arrow[rr, "h"] \arrow[dr] & & T_E \arrow[urr, "\pi_*"] \arrow[dl] & M & \\
 & E \arrow[urr, "\pi"'] & & &
\end{tikzcd}$$

commutes and its faces

$$\begin{tikzcd}
L \times_M E \arrow[r, "h"] \arrow[d] & T_E \arrow[d] \\
E \arrow[r, "\mathrm{id}"'] & E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
L \times_M E \arrow[r, "h"] \arrow[d] & T_E \arrow[d, "\pi_*"] \\
L \arrow[r, "\rho"'] & T_M
\end{tikzcd}$$

are vector bundle maps.

Definition 3. *A vector field X on E is said to be projectable onto M if $\pi(e_1) = \pi(e_2)$ implies $\pi_*(X_{e_1}) = \pi_*(X_{e_2})$. By $\mathfrak{X}_\pi(E)$, we denote the space of vector fields X on E which are projectable onto M and whose flow $\Phi_t^X : E \to E$ is a vector bundle automorphism over the flow $\Phi_t^{\pi_* X} : M \to M$ of the projected vector field $\pi_* X$ on M. The space $\mathfrak{X}_\pi(E)$ is obviously a module over the ring $C^\infty(M)$. A (linear) L-connection on E is a morphism of $C^\infty(M)$-modules $\Gamma(L) \xrightarrow{H} \mathfrak{X}_\pi(E)$, called horizontal distribution, such that the diagram*

$$\begin{tikzcd}
\Gamma(L) \arrow[rr, "H"] \arrow[dr, "\rho"'] & & \mathfrak{X}_\pi(E) \arrow[dl, "\pi_*"] \\
 & \mathfrak{X}(M) &
\end{tikzcd}$$

commutes.

Covariant differential, covariant derivative, horizontal lift, and horizontal distribution are related to one another by the identities

$$\nabla_l e = \langle d^\nabla e, l\rangle;$$
$$e_*\rho(l_x) - h(l_x, e_x) = \tau_{e_x}(\nabla_l e)_x;$$
$$H(l)|_{e_x} = h(l_x, e_x),$$

which hold for all $x \in M$, $l \in \Gamma(L)$, and $e \in \Gamma(E)$. Here in the second equation, τ_{e_x} denotes the canonical isomorphism between the fiber E_x and its tangent space at the point e_x. This second equation can be rewritten as

$$h(l_x, e_x) f_\nu = \langle \nabla_{l_x}\nu, e_x\rangle, \tag{1}$$

where f_ν denotes the fiberwise linear function on E determined by $\nu \in \Gamma(E^*)$.

The following assertions are equivalent:

$$\nabla_{l_1}\nabla_{l_2} - \nabla_{l_2}\nabla_{l_1} = \nabla_{[l_1,l_2]};$$
$$H([l_1, l_2]) = [H(l_1), H(l_2)].$$

When they are satisfied for all $l_1, l_2 \in \Gamma(L)$, the connection is said to be flat. An L-module is a vector bundle $E \to M$ endowed with a flat (linear) L-connection. A flat (linear) L-connection will also be called an L-action or L-representation. When the L-connection on E is flat, $(d^\nabla)^2 = 0$ and $(\Gamma(\wedge^\bullet L \otimes E), d^\nabla)$ is a cochain complex, whose cohomology groups $H^\bullet(L; E)$ are the so-called Lie algebroid cohomology groups of L with values in E.

By a *Lie pair* (L, A), we mean a Lie algebroid L over a manifold M and a subalgebroid A of L.

Proposition 4. *The quotient L/A of a Lie pair (L, A) is an A-module; the action of A on L/A is defined by*

$$\nabla_a\big(q(l)\big) = q([a, l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L),$$

where q denotes the projection $L \to L/A$. Being dual to L/A, the annihilator $A^\perp$ of A in L^* is also an A-module.

Assume now that A and B are two Lie subalgebroids of a Lie algebroid L such that L and $A \oplus B$ are isomorphic as vector bundles. Then $L/A \cong B$ is naturally an A-module while $L/B \cong A$ is naturally a B-module. The Lie algebroids A and B are said to form a matched pair.

Definition 5 ([31, 34, 32])**.** *Two (real or complex) Lie algebroids A and B over the same base manifold M and with respective anchors ρ_A and ρ_B are said to form a matched pair if there exists an action ∇ of A on B and an action Δ of B on A such that the identities*

$$[\rho_A(X), \rho_B(Y)] = -\rho_A\big(\Delta_Y X\big) + \rho_B\big(\nabla_X Y\big),$$
$$\nabla_X[Y_1, Y_2] = [\nabla_X Y_1, Y_2] + [Y_1, \nabla_X Y_2] + \nabla_{\Delta_{Y_2}X} Y_1 - \nabla_{\Delta_{Y_1}X} Y_2,$$
$$\Delta_Y[X_1, X_2] = [\Delta_Y X_1, X_2] + [X_1, \Delta_Y X_2] + \Delta_{\nabla_{X_2}Y} X_1 - \Delta_{\nabla_{X_1}Y} X_2,$$

hold for all $X_1, X_2, X \in \Gamma(A)$ and $Y_1, Y_2, Y \in \Gamma(B)$.

Proposition 6 ([34, 32])**.** *Given a matched pair (A, B) of Lie algebroids, there is a Lie algebroid structure $A \bowtie B$ on the direct sum vector bundle $A \oplus B$, with anchor*

$$X \oplus Y \mapsto \rho_A(X) + \rho_B(Y)$$

and bracket

$$[X_1 \oplus Y_1, X_2 \oplus Y_2] = \big([X_1, X_2] + \Delta_{Y_1}X_2 - \Delta_{Y_2}X_1\big) \oplus \big([Y_1, Y_2] + \nabla_{X_1}Y_2 - \nabla_{X_2}Y_1\big).$$

Conversely, if $A \oplus B$ has a Lie algebroid structure for which $A \oplus 0$ and $0 \oplus B$ are Lie subalgebroids, then the representations ∇ and Δ defined by

$$[X \oplus 0, 0 \oplus Y] = -\Delta_Y X \oplus \nabla_X Y$$

endow the couple (A, B) with a structure of matched pair.

Example 7. *A Lie algebra is a Lie algebroid whose base manifold is the one-point space. If the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ of a vector space $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ is endowed with a Lie algebra structure such that the direct summands $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras and*

$$[X, \alpha] = \operatorname{ad}^*_X \alpha - \operatorname{ad}^*_\alpha X, \qquad \forall X \in \mathfrak{g}, \alpha \in \mathfrak{g}^*,$$

the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is said to be a Lie bialgebra. Lie bialgebras are instances of matched pairs of Lie algebroids.

Example 8. *Let X be a complex manifold. Then $(T_X^{0,1}, T_X^{1,0})$ is a matched pair, where the actions are given by*

$$\nabla_{X^{0,1}} X^{1,0} = \operatorname{pr}^{1,0}[X^{0,1}, X^{1,0}] \qquad \text{and} \qquad \Delta_{X^{1,0}} X^{0,1} = \operatorname{pr}^{0,1}[X^{1,0}, X^{0,1}],$$

for all $X^{0,1} \in \Gamma(T_X^{0,1})$ and $X^{1,0} \in \Gamma(T_X^{1,0})$. Hence $T_X^{0,1} \bowtie T_X^{1,0}$ and $T_X \otimes \mathbb{C}$ are isomorphic as complex Lie algebroids. More generally, given a holomorphic Lie algebroid A, the pair $(A^{0,1}, A^{1,0})$ is a matched pair of Lie algebroids and $A^{0,1} \bowtie A^{1,0}$ is isomorphic, as a complex Lie algebroid, to $A \otimes \mathbb{C}$.

Example 9. *Let D be an integrable distribution on a smooth manifold M. Then D is a Lie subalgebroid of T_X, and the normal bundle T_X/D is canonically a D-module. The D-action on T_X/D is usually called Bott connection* [8]*. Moreover, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two transversal foliations on a smooth manifold M, the corresponding tangent distributions $T_{\mathcal{F}_1}$ and $T_{\mathcal{F}_2}$ constitute a matched pair of Lie algebroids with $T_{\mathcal{F}_1} \bowtie T_{\mathcal{F}_2} \cong T_X$.*

Example 10. *Let G be a Poisson Lie group and let (P, π) be a Poisson G-space, i.e. a Poisson manifold together with a G-action defined by a Poisson map $G \times P \to P$. According to Lu* [31]*, $A = (T^*P)_\pi$ and $B = P \rtimes \mathfrak{g}$ form a matched pair of Lie algebroids.*

Remark 11. *A matched pair of Lie algebroids $L = A \bowtie B$ can be seen as a Lie pair (L, A) together with a splitting $j : B \to L$ of the short exact sequence $0 \to A \to L \to B \to 0$, whose image $j(B)$ happens to be a Lie subalgebroid of L.*

2. Atiyah classes

2.1. **Prelude: holomorphic connections.** The Atiyah class of a holomorphic vector bundle E over a complex manifold X is the obstruction class to the existence of a holomorphic (linear) connection. It is constructed in the following way. The vector bundle $\mathscr{J}^1 E$ of jets (of order 1) of holomorphic sections of $E \to X$ fits into the canonical short exact sequence of holomorphic vector bundles

$$0 \to T_X^* \otimes E \to \mathscr{J}^1 E \to E \to 0$$

over the complex manifold X. The Atiyah class of $E \to X$ is the extension class

$$\alpha_E \in \operatorname{Ext}^1_X(E, T_X^* \otimes E)$$

of this short exact sequence [3, 22].

There are canonical isomorphisms between the abelian groups $\operatorname{Ext}^1_X(E, T_X^* \otimes E)$ and $\operatorname{Hom}_{D^b(X)}(T_X \otimes E, E[1])$, the sheaf cohomology group $H^1(X, T_X^* \otimes \operatorname{End} E)$ and the Dolbeault cohomology group $H^{1,1}(X, \operatorname{End} E)$. A Dolbeault representative of the Atiyah class can be obtained in the following way. Considering $T_X^{1,0}$ as a complex Lie algebroid, choose a $T_X^{1,0}$-connection $\nabla^{1,0}$ on E. Being a holomorphic vector bundle, E carries a canonical

flat $T_X^{0,1}$-connection $\overline{\partial}$. Adding $\nabla^{1,0}$ and $\overline{\partial}$, we obtain a $T_X\otimes\mathbb{C}$-connection ∇ on E. The element $\mathcal{R}\in\Omega^{1,1}(\operatorname{End}E)$ defined by

$$\mathcal{R}(X^{0,1};Y^{1,0})s=\nabla_{X^{0,1}}\nabla_{Y^{1,0}}s-\nabla_{Y^{1,0}}\nabla_{X^{0,1}}s-\nabla_{[X^{0,1},Y^{1,0}]}s \tag{2}$$

is a Dolbeault 1-cocycle whose cohomology class (which is independent of the choice of $\nabla^{1,0}$) is the Atiyah class $\alpha_E\in H^{1,1}(X,\operatorname{End}E)$.

2.2. **Existence of A-compatible L-connections.**

2.2.1. *Extension of an A-action to a compatible L-connection.* Throughout this section, (L,A) is a Lie pair and E is an A-module. The symbols $\mathscr{E}$ and $\mathscr{L}$ will denote the sheaves on M defined by

$$\mathscr{E}(U)=\{e\in\Gamma(U;E)\text{ s.t. }\nabla_a e=0,\forall a\in\Gamma(U;A)\},$$

and

$$\mathscr{L}(U)=\{l\in\Gamma(U;L)\text{ s.t. }[a,l]\in\Gamma(U;A),\forall a\in\Gamma(U;A)\},$$

where U denotes an arbitrary open subset of M.

Lemma 12. *Given an A-module E, there always exists an L-connection on E extending the given A-connection. Moreover, if ∇^1 and ∇^2 are two such extensions, then $d^{\nabla^2}-d^{\nabla^1}\in\Gamma(A^\perp\otimes\operatorname{End}E)$, where $A^\perp$ denotes the annihilator of A in L^*.*

Proof. Choose a subbundle B of L such that $L=A\oplus B$ and a T_M-connection $\nabla^{(T_M)}$ on E — this is always possible. Then extend A-connection $\nabla^{(A)}$ to an L-connection $\nabla^{(L)}$ on E by setting

$$\nabla^{(L)}_{a+b}=\nabla^{(A)}_a+\nabla^{(T_M)}_{\rho(b)},$$

where ρ denotes the anchor map $L\to T_M$. The difference $l\mapsto\nabla^1_l-\nabla^2_l$ of two such extensions ∇^1 and ∇^2 is a bundle map $L\to\operatorname{End}E$, which vanishes on A. $\square$

Definition 13. *An L-connection ∇ on E is said to be A-compatible if (1) it extends the given A-action on E and (2) it satisfies*

$$\nabla_a\nabla_l-\nabla_l\nabla_a=\nabla_{[a,l]},\quad\forall a\in\Gamma(A),l\in\Gamma(L).$$

Proposition 14. *Let ∇ be an L-connection on E extending its A-action. Provided that the sheaf of smooth sections of E is isomorphic to $C^\infty_M\otimes_{\mathbb{R}}\mathscr{E}$ and the sheaf of smooth sections of L is isomorphic to $C^\infty_M\otimes_{\mathbb{R}}\mathscr{L}$, the L-connection ∇ is A-compatible if and only if $\nabla_{\mathscr{L}}\mathscr{E}\subset\mathscr{E}$.*

Proof. For any $a\in\Gamma(U;A)$, $l\in\mathscr{L}(U)$ and $e\in\mathscr{E}(U)$, we have $[a,l]\in\Gamma(U;A)$, $\nabla_a e=0$, and $\nabla_{[a,e]}=0$ so that, if ∇ is A-compatible, we obtain

$$\nabla_a\nabla_l e=\nabla_a\nabla_l e-\nabla_l\nabla_a e-\nabla_{[a,l]}e=0.$$

Hence $\nabla_l e\in\mathscr{E}(U)$ and this proves that $\nabla_{\mathscr{L}}\mathscr{E}\subset\mathscr{E}$. Conversely, if $\nabla_{\mathscr{L}}\mathscr{E}\subset\mathscr{E}$, then

$$\big(\nabla_a\nabla_{f\cdot l}-\nabla_{f\cdot l}\nabla_a-\nabla_{[a,f\cdot l]}\big)(g\cdot e)=fg\cdot\nabla_a\nabla_l e=0,$$

for all $a\in\Gamma(U;A)$, $f,g\in C^\infty_U$, $l\in\mathscr{L}(U)$ and $e\in\mathscr{E}(U)$. Since $\Gamma(U;E)=C^\infty_U\otimes_{\mathbb{R}}\mathscr{E}$, it follows that ∇ is A-compatible. $\square$

Remark 15. *Given a matched pair of Lie algebroids (A,B) and an A-module E, consider the Lie algebroid $L=A\bowtie B$. An A-compatible L-connection on E determines a B-connection on E satisfying*

$$\nabla_a\nabla_b e-\nabla_b\nabla_a e=\nabla_{[a,b]}e,\quad\forall a\in\Gamma(A),b\in\Gamma(B),e\in\Gamma(E).$$

The converse is also true.

2.2.2. *Atiyah class: obstruction to compatibility.* Assume (L, A) is a Lie pair, E is an A-module, and ∇ is an L-connection on E extending its A-action.

The curvature of ∇ is the bundle map $R^{\nabla} : \wedge^2 L \to \operatorname{End} E$ defined by

$$R^{\nabla}(l_1, l_2) = \nabla_{l_1}\nabla_{l_2} - \nabla_{l_2}\nabla_{l_1} - \nabla_{[l_1,l_2]}, \quad \forall\, l_1, l_2 \in \Gamma(L).$$

Since E is an A-module, its restriction to $\wedge^2 A$ vanishes. Hence the curvature induces a section $R_E^{\nabla} \in \Gamma(A^* \otimes A^{\perp} \otimes \operatorname{End} E)$ or, equivalently, a bundle map $R_E^{\nabla} : A \otimes (L/A) \to \operatorname{End} E$ given by

$$R_E^{\nabla}\big(a; q(l)\big) = R^{\nabla}(a, l) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \quad \forall a \in \Gamma(A), l \in \Gamma(L). \tag{3}$$

The L-connection ∇ is compatible with the A-module structure of E if and only if $R_E^{\nabla} = 0$.

Theorem 16. (a) *The section R_E^{∇} of $A^* \otimes A^{\perp} \otimes \operatorname{End} E$ is a 1-cocycle for the Lie algebroid A with values in the A-module $A^{\perp} \otimes \operatorname{End} E$. We call R_E^{∇} the Atiyah cocycle associated with the L-connection ∇ that extends the A-module structure of E.*

(b) *The cohomology class $\alpha_E \in H^1(A; A^{\perp} \otimes \operatorname{End} E)$ of the cocycle R_E^{∇} does not depend on the choice of L-connection extending the A-action and is called the* Atiyah class *of the A-module E.*

(c) *The Atiyah class α_E of E vanishes if and only if there exists an A-compatible L-connection on E.*

Proof. We use the symbol ∂^A to denote the covariant differential associated to the action of the Lie algebroid A on $A^{\perp} \otimes \operatorname{End} E$.
(a) The differential Bianchi identity states that $d^{\nabla} R^{\nabla} : \wedge^3 L \to \operatorname{End} E$ is identically zero. Thus, for any $a_1, a_2 \in \Gamma(A)$, $l \in \Gamma(L)$, we have

$$\begin{aligned}
0 =& (d^{\nabla} R^{\nabla})(a_1, a_2, l) \\
=& \nabla_{a_1}(R^{\nabla}(a_2, l)) - \nabla_{a_2}(R^{\nabla}(a_1, l)) + \nabla_l(R^{\nabla}(a_1, a_2)) \\
& - R^{\nabla}([a_1, a_2], l) + R^{\nabla}([a_1, l], a_2) - R^{\nabla}([a_2, l], a_1) \\
=& \nabla_{a_1}\big(R_E^{\nabla}(a_2; q(l))\big) - \nabla_{a_2}\big(R_E^{\nabla}(a_1; q(l))\big) \\
& - R_E^{\nabla}([a_1, a_2]; q(l)) - R_E^{\nabla}(a_2; \nabla_{a_1} q(l)) + R_E^{\nabla}(a_1; \nabla_{a_2} q(l)) \\
=& \Big(\nabla_{a_1}\big(R_E^{\nabla}(a_2; q(l))\big) - R_E^{\nabla}\big(a_2; \nabla_{a_1} q(l)\big)\Big) \\
& - \Big(\nabla_{a_2}\big(R_E^{\nabla}(a_1; q(l))\big) - R_E^{\nabla}\big(a_1; \nabla_{a_2} q(l)\big)\Big) - R_E^{\nabla}\big([a_1, a_2]; q(l)\big) \\
=& \big(\partial^A R_E^{\nabla}\big)(a_1, a_2; q(l)).
\end{aligned}$$

Therefore $\partial^A R_E^{\nabla} = 0$.
(b) By Lemma 12, if ∇^1 and ∇^2 are two L-connections that extend the A-action, then $\nabla_l^1 - \nabla_l^2 = \phi(l)$ for some $\phi \in \Gamma(A^{\perp} \otimes \operatorname{End} E)$, and

$$\begin{aligned}
& R_E^{\nabla_1}(a; q(l)) \cdot e - R_E^{\nabla_2}(a; q(l)) \cdot e \\
&\quad = \nabla_a(\nabla_l^1 - \nabla_l^2)e - (\nabla_l^1 - \nabla_l^2)\nabla_a e - (\nabla_{[a,l]}^1 - \nabla_{[a,l]}^2)e \\
&\quad = \nabla_a\big(\phi(l) \cdot e\big) - \phi(l) \cdot (\nabla_a e) - \phi([a, l])e \\
&\quad = \big(\partial^A \phi\big)(a; l) \cdot e.
\end{aligned}$$

So $R_E^{\nabla_1} - R_E^{\nabla_2} = \partial^A \phi$.
(c) It is clear that R_E^{∇} vanishes if and only if ∇ is A-compatible. Now, if $R_E^{\nabla} = \partial^A \phi$ for some $\phi \in \Gamma(A^{\perp} \otimes \operatorname{End} E)$, set $\nabla' = \nabla - \phi$. Then $R_E^{\nabla'} = 0$, which implies that ∇' is A-compatible. $\square$

Remark 17. *When the Lie pair (L, A) is a matched pair of Lie algebroids, i.e. $L = A \bowtie B$, our definition of Atiyah class is a special case of the Atiyah class of a dDG algebra developed by Calaque and Van den Bergh* [12]. *Hence in the matched pair case, Theorem 16 (a)-(b) is a consequence of Lemma 8.2.4 in* [12].

Example 18. *Let X be a complex manifold and E a holomorphic vector bundle over X. Then $A = T_X^{0,1}$ and $B = T_X^{1,0}$ form a matched pair of Lie algebroids and $L = A \bowtie B$ is isomorphic to $T_X \otimes \mathbb{C}$. Moreover E is an A-module* [29]. *It is simple to see that holomorphic T_X-connections on E are equivalent to L-connections on E compatible with the A-action (as well as to A-compatible B-connections on E — see Remark 15). In this case, the Atiyah cocycle is exactly the Dolbeault 1-cocycle $\mathcal{R}$ defined by Equation* (2).

Example 19. *A holomorphic Lie algebroid K over a complex manifold X yields a matched pair of complex Lie algebroids $(T_X^{0,1}, K^{1,0})$* [29]. *The Atiyah class of the $T_X^{0,1}$-module $K^{1,0}$ is the Atiyah class for K studied extensively by Calaque and Van den Bergh in* [12].

Example 20. *In* [35], *Molino introduced an Atiyah class for connections "transversal to a foliation," which measures the obstruction to their "projectability." Although not phrased in the language of Lie algebroids, his construction is a special case of ours. Here L is the tangent bundle T_M, A is the tangent bundle to a foliation $\mathcal{F}$ of M, and the A-module E is a vector bundle on M foliated over $\mathcal{F}$. A transversal connection is an L-connection on E which extends the A-action. It is said to be projectable precisely if it is A-compatible, i.e. if it is preserved by parallel transport along any path tangent to $\mathcal{F}$.*

Example 21. *Let $\mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{d}$. Given an $\mathfrak{g}$-module E (i.e. a Lie algebra morphism $\boldsymbol{A} : \mathfrak{g} \to \operatorname{End} E$), and a $\mathfrak{d}$-connection on E extending it (i.e. a linear map $\boldsymbol{L} : \mathfrak{d} \to \operatorname{End} E$ whose restriction to $\mathfrak{g}$ is $\boldsymbol{A}$), the Atiyah class is the element in the Chevalley-Eilenberg cohomology group $H^1(\mathfrak{g}; \mathfrak{g}^\perp \otimes \operatorname{End}(E))$ determined by $\partial^{\mathfrak{g}} \boldsymbol{L}$. (The symbol $\partial^{\mathfrak{g}}$ denotes the Chevalley-Eilenberg coboundary of $\mathfrak{d}^* \otimes \operatorname{End}(E)$-valued $\mathfrak{g}$-cochains.) Here $\boldsymbol{L}$ is considered as an element in $\mathfrak{d}^* \otimes \operatorname{End}(E)$, which is, in general, not in $\mathfrak{g}^\perp \otimes \operatorname{End}(E)$. Hence, in general, $\partial^{\mathfrak{g}} \boldsymbol{L}$ does not vanish in $H^1(\mathfrak{g}; \mathfrak{g}^\perp \otimes \operatorname{End}(E))$.*

The following example is due to Calaque-Căldăraru-Tu [10].

Example 22. *Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its standard basis*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

Together, the matrices h and e generate the Lie subalgebra $\mathfrak{g}$ of 2×2 traceless upper triangular matrices. We identify the quotient $\mathfrak{sl}_2(\mathbb{C})/\mathfrak{g}$ to the nilpotent Lie subalgebra $\mathfrak{n}$ generated by f. Note that $\mathfrak{g}$ and $\mathfrak{n}$ form a matched pair of Lie algebras with sum $\mathfrak{g} \oplus \mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$. The bilinear map $\theta : \mathfrak{n} \otimes \mathfrak{n} \to \mathfrak{n}$ defined by $\theta(f, f) = f$ is a generator of the one-dimensional $\mathfrak{g}$-module $\mathfrak{g}^\perp \otimes \operatorname{End}(\mathfrak{n}) \cong \operatorname{Hom}(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$. The action of $\mathfrak{g}$ on $\operatorname{Hom}(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ is given by $h \cdot \theta = 2\theta$ and $e \cdot \theta = 0$. One checks that the degree 1 cohomology $H^1(\mathfrak{g}; \mathfrak{g}^\perp \otimes \operatorname{End}(\mathfrak{n}))$ is a one-dimensional vector space generated by the Atiyah class $\alpha_{\mathfrak{n}}$ of the $\mathfrak{g}$-module $\mathfrak{n}$.

2.3. **Functoriality.** Let M and N be smooth manifolds, $f : N \to M$ be a smooth map, A be a Lie algebroid over M with anchor $\rho : A \to TM$, and E be a smooth vector bundle over M.

Let f^*E denote the pullback of E through f, i.e. the fibered product of N and E over M:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ N \arrow[r, "f"'] & M. \end{tikzcd}$$

If the anchor ρ and the differential of f are transversal (i.e. $f_*(TN) + \rho(A) = TM|_N$), we can consider the fibered product $f^\star A$ of TN and A over TM:

$$\begin{tikzcd} f^\star A \arrow[r] \arrow[d] & A \arrow[d, "\rho"] \\ TN \arrow[r, "f_*"'] & TM. \end{tikzcd}$$

Note that ρ and f_* are automatically transversal when f is a surjective submersion or when the Lie algebroid A is transitive. It is clear that $f^\star A$ is a vector bundle over N. However, note that $f^\star A \neq f^*A$. The fiber of $f^\star A$ over a point $n \in N$ is

$$(f^\star A)_n = \left\{ (x, a) \in T_nN \oplus A_{f(n)} \middle| f_*(x) = \rho(a) \right\}.$$

The Lie algebroid structure on A induces a Lie algebroid structure on $f^\star A \to N$; its anchor is the projection $f^\star A \to TN$ and its bracket is given by

$$[(x_1, a_1), (x_2, a_2)] = ([x_1, x_2], [a_1, a_2]),$$

for any $x_1, x_2 \in \mathfrak{X}(N)$ and $a_1, a_2 \in \Gamma(A)$ such that $f_*(x_1) = \rho(a_1)$ and $f_*(x_2) = \rho(a_2)$ (see [19] for details).

Proposition 23. *Let A be a Lie algebroid over M and $f : N \to M$ a smooth map whose differential $f_* : TN \to TM$ is transversal to the anchor of A. Then*

(a) *If E is a module over A, then f^*E is a module over $f^\star A$.*

(b) *The map f induces a natural homomorphism*

$$f^\dagger : H^\bullet(A; E) \to H^\bullet(f^\star A; f^*E).$$

Proof. The first assertion is easily proved if one thinks of Lie algebroid modules in terms of horizontal lifting. The second assertion follows from a direct verification. □

The following proposition is immediate.

Proposition 24. *If (L, A) is a Lie pair over M, and $f : N \to M$ a smooth map whose differential $f_* : TN \to TM$ is transversal to the anchor of A, then $(f^\star L, f^\star A)$ is a Lie pair over N.*

Given a Lie pair (L, A) over a smooth manifold M and a smooth map $f : N \to M$ whose differential $f_* : TN \to TM$ is transversal to the anchor of A (otherwise $f^\star A$ and $f^\star L$ could be singular), there is a canonical morphism of vector bundles $f^\star L \to f^*(L/A)$ over N:

$$f^\star L \ni (x_n, a_{f(n)}) \mapsto a_{f(n)} + A_{f(n)} \in f^* \left(\frac{L}{A} \right),$$

whose kernel is exactly $f^\star A$. In other words, we have an exact sequence of vector bundles

$$0 \to f^\star A \to f^\star L \to f^*(L/A).$$

Therefore $f^\star L / f^\star A$ can be seen as a vector subbundle of $f^*(L/A)$.

Lemma 25. *Under the hypothesis of Proposition 24, the inclusion*

$$I : \frac{f^\star L}{f^\star A} \to f^*\left(\frac{L}{A}\right)$$

intertwines the $f^\star A$-module structures of $(f^\star L)/(f^\star A)$ and $f^*(L/A)$.

Dualizing it, as a consequence, we obtain the epimorphism of vector bundles

$$I^\dagger : f^*(A^\perp) \to (f^\star A)^\perp,$$

which is a morphism of $f^\star A$-modules. Note that, when f is a surjective submersion, I is surjective and thus both I and $I^\dagger$ are isomorphisms of $f^\star A$-modules.

We are now ready to state the main result in this subsection.

Theorem 26. *Let (L, A) be a Lie pair over M, and $f : N \to M$ a smooth map whose differential $f_* : TN \to TM$ is transversal to the anchor of A. Assume that E is an A-module. Then the composition of homomorphisms*

$$H^1\big(A; A^\perp \otimes \operatorname{End} E\big) \xrightarrow{f^\dagger} H^1\big(f^\star A; f^*(A^\perp \otimes \operatorname{End} E)\big) \xrightarrow{I^\dagger} H^1\big(f^\star A; (f^\star A)^\perp \otimes \operatorname{End}(f^* E)\big)$$

maps the Atiyah class of E relative to the Lie pair (L, A) onto the Atiyah class of f^*E relative to the Lie pair $(f^\star L, f^\star A)$:

$$(I^\dagger \circ f^\dagger)(\alpha_E) = \alpha_{f^*E}.$$

2.4. **Scalar Atiyah classes and Todd class.** Let (L, A) be a Lie pair. We define the scalar Atiyah classes [3] of an A-module E by

$$c_k(E) := \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \operatorname{tr}\left(\alpha_E^k\right) \in H^k(A; \wedge^k A^\perp).$$

Here α_E^k denotes the image of $\alpha_E \otimes \cdots \otimes \alpha_E$ under the natural map

$$H^1(A; A^\perp \otimes \operatorname{End} E) \times \cdots \times H^1(A; A^\perp \otimes \operatorname{End} E) \to H^k(A; \wedge^k A^\perp \otimes \operatorname{End} E)$$

induced by the composition in $\operatorname{End} E$ and the wedge product in $\wedge^\bullet A^\perp$.

If E is a holomorphic vector bundle over a compact Kähler manifold X, the natural inclusion of $H^k(X, \Omega^k)$ into $H^{2k}(X, \mathbb{C})$ maps the scalar Atiyah classes of E relative to the Lie pair $(L = T_X \otimes \mathbb{C}, A = T_X^{0,1})$ to the Chern classes of E.

The Todd class of the A-module E relative to the Lie pair (L, A) is the cohomology class

$$\operatorname{Td}(E) = \det\left(\frac{\alpha_E}{1 - e^{-\alpha_E}}\right) \in H^\bullet(A; \wedge^\bullet A^\perp).$$

The following propositions can be verified directly.

Proposition 27. *Let (L, A) be a Lie pair and let E_1, E_2 be A-modules. Then*

$$\operatorname{Td}(E_1 \oplus E_2) = \operatorname{Td} E_1 \cdot \operatorname{Td} E_2$$

Proposition 28. *Under the hypothesis of Theorem 26, we have*

$$c_k(f^*E) = (I^\dagger \circ f^\dagger)(c_k E) \in H^k\big(f^\star A; \wedge^k (f^\star A)^\perp\big)$$
$$\operatorname{Td}(f^*E) = (I^\dagger \circ f^\dagger)(\operatorname{Td} E) \in H^\bullet\big(f^\star A; \wedge^\bullet (f^\star A)^\perp\big)$$

2.5. **Jet short exact sequence.**

2.5.1. *The jet bundle* $\mathscr{J}^1_{L/A}E$. Let M be a smooth manifold, let $L \to M$ be a Lie algebroid, and let $E \xrightarrow{\pi} M$ be a vector bundle.

An L-jet (of order 1) on E (at $e_x \in E$) is a linear map $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ such that the diagram

$$\begin{tikzcd} L_{\pi(e_x)} \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & T_{e_x}E \arrow[dl, "\pi_{*e_x}"] \\ & T_{\pi(e_x)}M & \end{tikzcd}$$

commutes. The jet space $\mathscr{J}^1_L E$ is the manifold whose points are L-jets on E. It is a vector bundle over X: the projection $\mathscr{J}^1_L E \to M$ maps $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ to $\pi(e_x)$. It fits into the short exact sequence of vector bundles over M:

$$0 \longrightarrow L^* \otimes E \xrightarrow{\hat{f}} \mathscr{J}^1_L E \xrightarrow{\hat{g}} E \longrightarrow 0. \tag{4}$$

The surjection $\hat{g}$ maps $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ to e_x while the injection $\hat{f}$ maps $L_x \xrightarrow{\psi} E_x$ to $L_x \xrightarrow{\rho\oplus\psi} T_xM \oplus E_x \cong T_{0_x}E$.

Proposition 29. *A splitting* $s : E \to \mathscr{J}^1_L E$ *of the short exact sequence of vector bundles* (4) *determines a (linear)* L*-connection on* E*. The converse is also true.*

In general, there is no canonical choice of splitting for (4). However, the induced short exact sequence

$$0 \longrightarrow \Gamma(L^* \otimes E) \xrightarrow{\hat{f}_\sharp} \Gamma(\mathscr{J}^1_L E) \xrightarrow{\hat{g}_\sharp} \Gamma(E) \longrightarrow 0 \tag{5}$$

at the level of spaces of smooth sections splits canonically: if e is a section of E, then $\sigma e := e_* \circ \rho$ is a section of $\mathscr{J}^1_L E$ such that $\hat{g}_\sharp(\sigma e) = e$.

We note that the covariant differential $d^\nabla : \Gamma(E) \to \Gamma(L^* \otimes E)$ associated to a splitting $s : E \to \mathscr{J}^1_L E$ of the short exact sequence (4) is given by

$$\hat{f}_\sharp(d^\nabla e) = \sigma e - s_\sharp(e), \quad \forall e \in \Gamma(E).$$

Now assume A is a Lie subalgebroid of L and E is an A-module. The symbol h will denote the horizontal lifting associated to the A-action on E.

An h-extending L-jet (of order 1) on E is a linear map $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ such that the diagram

$$\begin{tikzcd} & A_{\pi(e_x)} \arrow[dl] \arrow[dr, "h(-{,}e_x)"] & \\ L_{\pi(e_x)} \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & T_{e_x}E \arrow[dl, "\pi_{*e_x}"] \\ & T_{\pi(e_x)}M & \end{tikzcd}$$

commutes. The jet space $\mathscr{J}^1_{L/A}E$ is the manifold whose points are h-extending L-jets on E. It is a vector bundle over M: the projection $\mathscr{J}^1_{L/A}E \to M$ maps $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ to $\pi(e_x)$.

Example 30. *When E is a holomorphic vector bundle over a complex manifold X, $A = T_X^{0,1}$ and $L = T_X \otimes \mathbb{C}$, the jet bundle $\mathscr{J}^1_{L/A}E$ is simply the bundle of jets (of order 1) of holomorphic sections of E.*

Consider the surjective morphism of vector bundles $\breve{g} : \mathscr{J}^1_{L/A}E \to E$, which maps $L_{\pi(e_x)} \xrightarrow{\phi} T_{e_x}E$ to e_x. Since $T_{0_x}E$ is canonically isomorphic to $T_xM \oplus E_x$, the kernel of $\breve{g}$ can be identified naturally to the subbundle K of $L^* \otimes E \to M$ consisting of all linear maps $L_x \xrightarrow{\psi} E_x$ which satisfy

$$h(a_x, 0_x) = \rho(a_x) + \psi(a_x), \quad \forall x \in M, a_x \in A_x.$$

Since the A-connection h on E is linear, $h(a_x, 0_x)$ must be the image of $\rho(a_x)$ under the differential of the zero section $M \xrightarrow{0} E$. Therefore, a linear map $L_x \xrightarrow{\psi} E_x$ is an element of K if and only if $\psi(a_x) = 0$ for all $a_x \in A$, so that $K = A^\perp \otimes E$. Hence we obtain the short exact sequence of vector bundles

$$0 \longrightarrow A^\perp \otimes E \xrightarrow{\breve{f}} \mathscr{J}^1_{L/A}E \xrightarrow{\breve{g}} E \longrightarrow 0, \tag{6}$$

where the injection $\breve{f}$ maps $L_x \xrightarrow{\psi} E_x$ to the jet

$$L_x \to T_{0_x}E \cong T_xM \oplus E_x \qquad l_x \mapsto \rho(l_x) + \psi(l_x).$$

In general, there is no canonical choice of splitting for (6).

Proposition 31. *A splitting $s : E \to \mathscr{J}^1_{L/A}E$ of the short exact sequence of vector bundles* (6) *determines a (linear) L-connection on E extending the A-action h. The converse is also true.*

Obviously, we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^\perp \otimes E \arrow[r, "\breve{f}"] \arrow[d] & \mathscr{J}^1_{L/A}E \arrow[r, "\breve{g}"] \arrow[d] & E \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & L^* \otimes E \arrow[r, "\hat{f}"] & \mathscr{J}^1_{L}E \arrow[r, "\hat{g}"] & E \arrow[r] & 0,
\end{tikzcd}$$

all of whose vertical arrows denote inclusions.

2.5.2. *An equivalent description of the jet bundle.*

Proposition 32. *An L-jet (of order 1) on E extending the A-action ∇ is a pair (D_x, e_x) consisting of a linear map $D_x : \Gamma(E^*) \to L_x^*$ and a point e_x in the fiber of E over $x \in M$, satisfying*

$$\langle D_x(\varepsilon), a_x \rangle = \langle \nabla_{a_x}\varepsilon, e_x \rangle \quad (\text{or equivalently} \quad D_x(\varepsilon) = \langle d^\nabla \varepsilon, e_x \rangle); \tag{7}$$

$$D_x(f\varepsilon) = f(x) \cdot D_x(\varepsilon) + \langle \varepsilon_x, e_x \rangle \cdot \rho^*(df), \tag{8}$$

for all $a_x \in A_x$, $\varepsilon \in \Gamma(E^*)$, and $f \in C^\infty(M)$.

Proof. Given such a pair (D_x, e_x), each $l_x \in L_x$ determines uniquely a tangent vector $\tau_x \in T_{e_x}E$ through the relations

$$\tau_x(\pi^* f) = \rho(l_x)f = \langle \rho^*(df), l_x \rangle \qquad \text{and} \qquad \tau_x(f_\varepsilon) = \langle D_x(\varepsilon), l_x \rangle,$$

where $\varepsilon \in \Gamma(E^*)$, $f \in C^\infty(M)$, and $f_\varepsilon \in C^\infty(E)$ is the fiberwise linear function associated to ε:

$$f_\varepsilon(e_x) = \langle \varepsilon_x, e_x \rangle.$$

Let $\phi_D : L_x \to T_{e_x}E$ be the map $l_x \mapsto \tau_x$. Clearly, ϕ_D is linear and satisfies $\pi_* \circ \phi_D = \rho$. Moreover, ϕ_D is an extension of the A-action since

$$\big(\phi_D(a_x)\big)(f_\varepsilon) = \langle D_x(\varepsilon), a_x\rangle = \langle \nabla_{a_x}\varepsilon, e_x\rangle = h(a_x, e_x)(f_\varepsilon).$$

Here we have made use of Equation (1). Hence $\phi_D \in \big(\mathscr{J}^1_{L/A}E\big)_x$ and $\breve{g}(\phi_D) = e_x$. Conversely, given an element $\phi : L_x \to T_{e_x}E$ of $\big(\mathscr{J}^1_{L/A}E\big)_x$ that projects to e_x under $\breve{g}$, we can define a linear map $D^\phi_x : \Gamma(E^*) \to L_x$ by the relation

$$\langle D^\phi_x(\varepsilon), l_x\rangle = \big(\phi(l_x)\big)(f_\varepsilon).$$

It is straightforward to check that (D^ϕ_x, e_x) satisfies (7) and (8). $\square$

Remark 33. *The surjection $\breve{g} : \mathscr{J}^1_{L/A}E \to E$ in* (6) *maps the 1-jet (D_x, e_x) to e_x. The injection $\breve{f}$ in* (6) *maps $\psi \in (A^\perp \otimes E)_x$ to $(\psi^\dagger_x, 0_x) \in \big(\mathscr{J}^1_{L/A}E\big)_x$, where 0_x is the zero vector of E_x and $\psi^\dagger_x : \Gamma(E^*) \to L^*_x$ is the linear map defined by*

$$\langle \psi^\dagger_x(\varepsilon), l_x\rangle = \langle \varepsilon_x, \psi(l_x)\rangle, \quad \forall l_x \in L_x, \varepsilon \in \Gamma(E^*).$$

Here ψ is considered as a linear map $L_x \to E_x$ whose kernel contains A_x.

2.5.3. *The jet bundle is an A-module.* The jet bundle $\mathscr{J}^1_{L/A}E$ can be naturally endowed with an A-action.

In the language of Proposition 32, a section of $\mathscr{J}^1_{L/A}E \to M$ consists of a section e of $E \to M$ and an $\mathbb{R}$-linear map $D : \Gamma(E^*) \to \Gamma(L^*)$ satisfying

$$\langle D(\varepsilon), a\rangle = \langle \nabla_a\varepsilon, e\rangle;$$
$$D(f\varepsilon) = f \cdot D(\varepsilon) + \langle \varepsilon, e\rangle \cdot \rho^*(df),$$

for all $f \in C^\infty(M)$, $\varepsilon \in \Gamma(E^*)$, and $a \in \Gamma(A)$.

Proposition 34. (a) *The jet bundle $\mathscr{J}^1_{L/A}E$ is a module over A; the covariant derivative*

$$\Gamma(A) \times \Gamma(\mathscr{J}^1_{L/A}E) \to \Gamma(\mathscr{J}^1_{L/A}E)$$

is given by $\nabla_a(D, e) = (\nabla_a D, \nabla_a e)$, where the $\mathbb{R}$-linear map $\nabla_a D : \Gamma(E^*) \to \Gamma(L^*)$ is defined as follows:

$$\langle \big(\nabla_a D\big)(\varepsilon), l\rangle = \rho(a)\langle D(\varepsilon), l\rangle - \langle D(\nabla_a\varepsilon), l\rangle - \langle D(\varepsilon), [a, l]\rangle. \tag{9}$$

(b) *Diagram* (6) *is a short exact sequence of A-modules.*

Proof. (a) Checking that (9) determines a connection is straightforward. The flatness of the A-connection on $\mathscr{J}^1_{L/A}E$ is a consequence of the flatness of the A-connection on E.
(b) By definition, $\breve{g}$ is a morphism of A-modules. Let us check that $\breve{f}$ is also a morphism of A-modules. For any $\psi \in \Gamma(A^\perp \otimes E)$, we have

$$\begin{aligned}\langle (\nabla_a\psi^\dagger)(\varepsilon), l\rangle &= \rho(a)\langle \psi^\dagger(\varepsilon), l\rangle - \langle \psi^\dagger(\nabla_a\varepsilon), l\rangle - \langle \psi^\dagger(\varepsilon), [a, l]\rangle \\ &= \rho(a)\langle \psi(l), \varepsilon\rangle - \langle \psi(l), \nabla_a\varepsilon\rangle - \langle \psi([a, l]), \varepsilon\rangle = \langle (\nabla_a\psi)(l), \varepsilon\rangle = \langle (\nabla_a\psi)^\dagger(\varepsilon), l\rangle. \quad \square\end{aligned}$$

2.5.4. *Alternative description of the A-action on $\mathscr{J}^1_{L/A}E$.* In this section, B will denote the quotient L/A of the Lie pair (L, A).

The proof of the following lemma is a tedious computation, which we omit.

Lemma 35. *The splitting $\sigma : \Gamma(E) \to \Gamma(\mathscr{J}^1_L E)$ of the short exact sequence (5) is not $C^\infty(M)$-linear. For every $e \in \Gamma(E)$ and $f \in C^\infty(M)$, we have*

$$\sigma(f \cdot e) - f \cdot \sigma e = \hat{f}_\sharp\big(\rho^*(df) \otimes e\big).$$

In general, σe need not be a section of $\mathscr{J}^1_{L/A}E$. Nevertheless, fixing a splitting of the short exact sequence of vector bundles

$$0 \longrightarrow A \xrightarrow{i} L \xrightarrow{q} B \longrightarrow 0\ ,$$

i.e. a pair of maps $j : B \to L$ and $p : L \to A$ such that $q \circ j = \mathrm{id}_B$, $p \circ i = \mathrm{id}_A$ and $i \circ p + j \circ q = \mathrm{id}_L$:

$$\begin{tikzcd} 0 \arrow[r, shift left] & A \arrow[l, shift left] \arrow[r, shift left, "i"] & L \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & B \arrow[l, shift left, "j"] \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}\ ,$$

naturally determines a splitting $\varsigma : \Gamma(E) \to \Gamma(\mathscr{J}^1_{L/A}E)$ of the short exact sequence of spaces of smooth sections

$$0 \longrightarrow \Gamma(A^\perp \otimes E) \xrightarrow{\breve{f}_\sharp} \Gamma(\mathscr{J}^1_{L/A}E) \xrightarrow{\breve{g}_\sharp} \Gamma(E) \longrightarrow 0$$

induced by (6). The image of $x \in M$ under the section ςe of $\mathscr{J}^1_{L/A}E$ associated to a section e of E by the splitting ς is the 1-jet

$$L_x \ni l_x \xmapsto{(\varsigma e)_x} h(p(l_x), e_x) + e_{*x}\big(\rho \circ j \circ q(l_x)\big) \in T_{e_x}E.$$

It is not difficult to see that $\breve{g}_\sharp(\varsigma e) = e$.

The proof of the following lemma is a tedious computation, which we omit.

Lemma 36. *The splitting $\varsigma : \Gamma(E) \to \Gamma(\mathscr{J}^1_{L/A}E)$ is not $C^\infty(M)$-linear. For every $e \in \Gamma(E)$ and $f \in C^\infty(M)$, we have*

$$\varsigma(f \cdot e) - f \cdot \varsigma e = \breve{f}_\sharp\Big(\big(l \mapsto \rho(j \circ q(l))f\big) \otimes e\Big).$$

Since both E and $A^\perp$ are modules over A, so is $A^\perp \otimes E$:

$$\langle \nabla_a(\lambda \otimes e), l\rangle = \rho(a)\big(\lambda(l)\big) \cdot e - \lambda([a, l]) \cdot e + \lambda(l) \cdot \nabla_a e,$$

where $a \in \Gamma(A)$, $\lambda \in \Gamma(A^\perp)$, $e \in \Gamma(E)$, and $l \in \Gamma(L)$.

Remark 37. *If $\mathscr{J}^1_{L/A}E$ was an A-module and $\breve{f} : A^\perp \otimes E \to \mathscr{J}^1_{L/A}E$ was a morphism of A-modules, then tedious computations would yield*

$$\varsigma(\nabla_{f\cdot a}e) - \nabla_{f\cdot a}(\varsigma e) = f \cdot \big(\varsigma(\nabla_a e) - \nabla_a(\varsigma e)\big) + \breve{f}_\sharp\Big(\big(l \mapsto \rho(j \circ q(l))f\big) \otimes \nabla_a e\Big)$$

and

$$\varsigma\big(\nabla_a(f \cdot e)\big) - \nabla_a\big(\varsigma(f \cdot e)\big) = f \cdot \big(\varsigma(\nabla_a e) - \nabla_a(\varsigma e)\big) + \breve{f}_\sharp\Big(\big(l \mapsto \rho(i \circ p[j \circ q(l), a])f\big) \otimes e\Big),$$

for any $a \in \Gamma(A)$, $e \in \Gamma(E)$, and $f \in C^\infty(M)$. Therefore, one cannot expect to define an A-action on the jet bundle $\mathscr{J}^1_{L/A}E$ simply by setting

$$\nabla_a\big(\breve{f}_\sharp(\lambda \otimes e)\big) = \breve{f}_\sharp\big(\nabla_a(\lambda \otimes e)\big);$$
$$\nabla_a(\varsigma e) = \varsigma(\nabla_a e).$$

The proof of the following Lemma is a tedious computation, which we omit.

Lemma 38. *Given $a \in \Gamma(A)$ and $e \in \Gamma(E)$, define $\Theta(a,e) \in \Gamma(A^\perp \otimes E)$ by the relation*

$$\langle \Theta(a,e), l\rangle = \nabla_{i\circ p[j\circ q(l),a]} e, \qquad \forall l \in \Gamma(L).$$

Then, for any $f \in C^\infty(M)$,

$$\Theta(f\cdot a, e) - f\cdot\Theta(a,e) = \big(l \mapsto \rho(j\circ q(l))f\big)\otimes \nabla_a e;$$
$$\Theta(a, f\cdot e) - f\cdot\Theta(a,e) = \big(l \mapsto \rho(i\circ p[j\circ q(l),a])f\big)\otimes e.$$

This suggests the following proposition.

Proposition 39. (a) *The jet bundle $\mathscr{J}^1_{L/A}E \to M$ is an A-module: the flat A-connection on $\mathscr{J}^1_{L/A}E$ is given by*

$$\nabla_a\big(\breve{f}_\sharp(\lambda\otimes e)\big) = \breve{f}_\sharp\big(\nabla_a(\lambda\otimes e)\big) \tag{10}$$
$$\nabla_a(\varsigma e) = \varsigma(\nabla_a e) - \breve{f}_\sharp\big(\Theta(a,e)\big), \tag{11}$$

for all $a \in \Gamma(A)$, $\lambda \in \Gamma(A^\perp)$, and $e \in \Gamma(E)$. Here $\Theta(a,e) \in \Gamma(A^\perp\otimes E)$ is defined by

$$\langle \Theta(a,e), l\rangle = \nabla_{p[j\circ q(l),a]} e, \qquad \forall l \in \Gamma(L).$$

(b) *Diagram* (6) *is a short exact sequence of A-modules.*

Sketch of proof. (a) It follows from Lemmas 36 and 38 that what we have defined is indeed a covariant derivative. Flatness follows from the Jacobi identity in $\Gamma(L)$ and the flatness of the A-connections on E and $A^\perp\otimes E$.
(b) It suffices to check that $\breve{g}$ is a morphism of A-modules. We have

$$\breve{g}_\sharp\big(\nabla_a(\varsigma e)\big) = \breve{g}_\sharp\big(\varsigma(\nabla_a e)\big) - \breve{g}_\sharp\breve{f}_\sharp\big(\Theta(a,e)\big) = \nabla_a e = \nabla_a\big(\breve{g}_\sharp(\varsigma e)\big). \qquad \square$$

Remark 40. *For a matched pair $L = A \bowtie B$, observe that $\Theta(a,e) \in \Gamma(B^*\otimes E)$ is given by the simple formula $\langle\Theta(a,e), b\rangle = \nabla_{\Delta_b a} e$.*

Proposition 41. *The A-actions on $\mathscr{J}^1_{L/A}E$ defined in Propositions 34 and 39 are identical.*

Proof. The $\mathbb{R}$-linear map $D^{\varsigma e} : \Gamma(E^*) \to \Gamma(L^*)$ determined by the section ςe of $\mathscr{J}^1_{L/A}E \to M$ (as explained in the proof of Proposition 32) satisfies

$$\langle D^{\varsigma e}(\varepsilon), a\rangle = \langle \nabla_a\varepsilon, e\rangle;$$
$$\langle D^{\varsigma e}(\varepsilon), j(b)\rangle = \Big(\varsigma e\big(j(b)\big)\Big)(f_\varepsilon) = \Big(e_*\circ\rho\big(j(b)\big)\Big)(f_\varepsilon) = \rho\big(j(b)\big)\langle\varepsilon, e\rangle,$$

for all $e \in \Gamma(E)$, $\varepsilon \in \Gamma(E^*)$, $a \in \Gamma(A)$, and $b \in \Gamma(B)$. The image of $D^{\varsigma e}$ under the action of $a \in \Gamma(A)$ is the $\mathbb{R}$-linear map $\nabla_a D^{\varsigma e} : \Gamma(E^*) \to \Gamma(L^*)$ defined by (9). We have

$$\begin{aligned}
&\langle\big(\nabla_{a'}D^{\varsigma e}\big)(\varepsilon), a\rangle \\
&\quad= \rho(a')\langle D^{\varsigma e}(\varepsilon), a\rangle - \langle D^{\varsigma e}(\nabla_{a'}\varepsilon), a\rangle - \langle D^{\varsigma e}(\varepsilon), [a',a]\rangle \\
&\quad= \rho(a')\langle\nabla_a\varepsilon, e\rangle - \langle\nabla_a\nabla_{a'}\varepsilon, e\rangle - \langle\nabla_{[a',a]}\varepsilon, e\rangle \\
&\quad= \rho(a')\langle\nabla_a\varepsilon, e\rangle - \langle\nabla_{a'}\nabla_a\varepsilon, e\rangle \\
&\quad= \langle\nabla_a\varepsilon, \nabla_{a'}e\rangle \\
&\quad= \langle D^{\varsigma(\nabla_{a'}e)}(\varepsilon), a\rangle
\end{aligned}$$

and

$$
\begin{aligned}
&\langle\big(\nabla_{a'}D^{\varsigma e}\big)(\varepsilon), j(b)\rangle \\
&\quad= \rho(a')\langle D^{\varsigma e}(\varepsilon), j(b)\rangle - \langle D^{\varsigma e}(\nabla_{a'}\varepsilon), j(b)\rangle - \langle D^{\varsigma e}(\varepsilon), [a', j(b)]\rangle \\
&\quad= \rho(a')\rho\big(j(b)\big)\langle\varepsilon, e\rangle - \rho\big(j(b)\big)\langle\nabla_{a'}\varepsilon, e\rangle \\
&\qquad - \langle D^{\varsigma e}(\varepsilon), -p[j(b), a'] + j(\nabla_{a'}b)\rangle \\
&\quad= \rho(a')\rho\big(j(b)\big)\langle\varepsilon, e\rangle - \rho\big(j(b)\big)\rho(a')\langle\varepsilon, e\rangle + \rho\big(j(b)\big)\langle\varepsilon, \nabla_{a'}e\rangle \\
&\qquad + \langle\nabla_{p[j(b),a']}\varepsilon, e\rangle - \rho\big(j(\nabla_{a'}b)\big)\langle\varepsilon, e\rangle \\
&\quad= \rho([a', j(b)])\langle\varepsilon, e\rangle + \rho\big(j(b)\big)\langle\varepsilon, \nabla_{a'}e\rangle + \rho(p[j(b), a'])\langle\varepsilon, e\rangle \\
&\qquad - \langle\varepsilon, \nabla_{p[j(b),a']}e\rangle - \rho\big(j(\nabla_{a'}b)\big)\langle\varepsilon, e\rangle \\
&\quad= \rho\big(j(b)\big)\langle\varepsilon, \nabla_{a'}e\rangle - \langle\varepsilon, \nabla_{p[j(b),a']}e\rangle \\
&\quad= \langle D^{\varsigma(\nabla_{a'}e)}(\varepsilon), j(b)\rangle - \langle\big(\Theta(a', e)\big)^{\dagger}(\varepsilon), j(b)\rangle.
\end{aligned}
$$

Therefore, we obtain

$$\nabla_{a'}D^{\varsigma e} = D^{\varsigma(\nabla_{a'}e)} - \big(\Theta(a', e)\big)^{\dagger},$$

which is equivalent to Equation (11). □

2.5.5. *The abelian category $\mathcal{A}$.* It is classical a result that the space $\Gamma(A)$ of smooth sections of a Lie algebroid A over a smooth manifold M is a Lie-Rinehart algebra over the commutative ring $C^\infty(M)$ [20, 38]. We denote the (abelian) category of modules over this Lie-Rinehart algebra by the symbol $\mathcal{A}$. Alternatively, the objects of $\mathcal{A}$ can be seen as left modules over the universal enveloping algebra $\mathcal{U}(A)$ of the Lie algebroid A. The space of smooth sections of an A-module, i.e. a vector bundle over M endowed with an A-action, is an object of $\mathcal{A}$.

We note that the derived category of $\mathcal{A}$, which we denote by $D(\mathcal{A})$, is a symmetric monoidal category [21]. The interchange isomorphism $\tau : X \otimes Y \to Y \otimes X$ of a pair of objects X and Y of $D(\mathcal{A})$ is given by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x. \tag{12}$$

2.5.6. *Extension class of the jet sequence.* A short exact sequence of A-modules

$$0 \longrightarrow P \xrightarrow{\alpha} Q \xrightarrow{\beta} R \longrightarrow 0 \tag{13}$$

determines an extension class in the group $\operatorname{Ext}^1_{\mathcal{A}}(R, P)$, which is naturally isomorphic to the Lie algebroid cohomology group $H^1(A; R^* \otimes P)$ [20].

Indeed, given a homomorphism of vector bundles $s : R \to Q$ such that $\beta \circ s = \mathrm{id}_R$, we have

$$s_\sharp(\nabla_a r) - \nabla_a\big(s_\sharp(r)\big) \in \ker\beta, \qquad \forall a \in \Gamma(A), r \in \Gamma(R)$$

so that the equation

$$s_\sharp(\nabla_a r) - \nabla_a\big(s_\sharp(r)\big) = \alpha_\sharp\big(\xi_s(a) \cdot r\big) \tag{14}$$

defines a vector bundle map $\xi_s : A \to \operatorname{Hom}(R, P)$. Rewriting (14) as

$$(\mathrm{id}_{R^*} \otimes \alpha) \circ \xi_s = \partial^A s \tag{15}$$

and recalling that $\alpha : P \to Q$ is a morphism of A-modules, we immediately see that

$$(\mathrm{id}_{R^*} \otimes \alpha) \circ \partial^A \xi_s = \partial^A\big((\mathrm{id}_{R^*} \otimes \alpha) \circ \xi_s\big) = \partial^A(\partial^A s) = 0.$$

Therefore $\partial^A \xi_s = 0$, i.e. ξ_s is a 1-cocycle for the Lie algebroid A with values in the A-module $R^* \otimes P$. It follows from Equation (15) that the cohomology class $[\xi_s] \in H^1(A; R^* \otimes P)$ of the 1-cocycle ξ_s defined by (14) is independent of the choice of the section $s : R \to Q$. In

fact, $[\xi_s]$ is the extension class in $\operatorname{Ext}^1_{\mathcal{A}}(R,P) \cong H^1(A;R^*\otimes P)$ of the short exact sequence of A-modules (13).

Proposition 42. *Given a Lie pair (L,A) and an A-module E, let ∇ denote the L-connection on E determined by a section $s: E \to \mathscr{J}^1_{L/A}E$ of the short exact sequence (6). When considered as sections of $A^*\otimes A^\perp\otimes \operatorname{End} E$, the bundle maps $\xi_s : A \to \operatorname{Hom}(E, A^\perp\otimes E)$ and $R^\nabla_E : A\otimes(L/A) \to \operatorname{End} E$ (respectively defined by (14) and (3)) are one and the same.*

Proof. Define $\breve{d}^\nabla : \Gamma(E) \to \Gamma(A^\perp\otimes E)$ by

$$\breve{f}_\sharp(\breve{d}^\nabla e) = \varsigma e - s_\sharp(e). \tag{16}$$

Then, for all $b\in B$, we have $\langle \breve{d}^\nabla e, j(b)\rangle = \nabla_{j(b)}e$.

For all $a\in\Gamma(A)$, and $e\in\Gamma(E)$, we have

$$\begin{aligned}
&\breve{f}_\sharp\big(\xi_s(a)\cdot e\big) \\
&\quad = s_\sharp(\nabla_a e) - \nabla_a(s_\sharp e) && \text{by (14),} \\
&\quad = \big(\varsigma(\nabla_a e) - \breve{f}_\sharp\breve{d}^\nabla(\nabla_a e)\big) - \big(\nabla_a(\varsigma e) - \nabla_a\breve{f}_\sharp(\breve{d}^\nabla e)\big) && \text{by (16),} \\
&\quad = \breve{f}_\sharp\big(\Theta(a,e) + \nabla_a(\breve{d}^\nabla e) - \breve{d}^\nabla(\nabla_a e)\big) && \text{by (10) and (11).}
\end{aligned}$$

Hence, for all $b\in\Gamma(B)$, we get

$$\begin{aligned}
&\langle \xi_s(a)\cdot e, j(b)\rangle \\
&\quad = \langle \Theta(a,e) + \nabla_a(\breve{d}^\nabla e) - \breve{d}^\nabla(\nabla_a e), j(b)\rangle \\
&\quad = \nabla_{p[j(b),a]}e + \nabla_a\langle \breve{d}^\nabla e, j(b)\rangle - \langle \breve{d}^\nabla e, j(\nabla_a b)\rangle - \langle \breve{d}^\nabla(\nabla_a e), j(b)\rangle \\
&\quad = \nabla_{p[j(b),a]}e + \nabla_a\nabla_{j(b)}e - \nabla_{j(\nabla_a b)}e - \nabla_{j(b)}\nabla_a e \\
&\quad = \nabla_a\nabla_{j(b)}e - \nabla_{j(b)}\nabla_a e - \nabla_{j(\nabla_a b)+p[a,j(b)]}e \\
&\quad = \nabla_a\nabla_{j(b)}e - \nabla_{j(b)}\nabla_a e - \nabla_{[a,j(b)]}e \\
&\quad = R^\nabla\big(a,j(b)\big)e \\
&\quad = R^\nabla_E(a;b)\cdot e.
\end{aligned}$$

This proves that $\xi_s = R^\nabla_E$. $\square$

Corollary 43. *Let (L,A) be a Lie pair, and E an A-module.*

(a) *A section $s: E\to \mathscr{J}^1_{L/A}E$ of the short exact sequence (6) is a morphism of A-modules if and only if the L-connection it induces on E is compatible with the A-action on E.*

(b) *The short exact sequence of A-modules (6) splits if and only if the Atiyah class α_E vanishes.*

Theorem 44. *Let (L,A) be a Lie pair, and E an A-module. The natural isomorphism*

$$\operatorname{Ext}^1_{\mathcal{A}}(E, A^\perp\otimes E) \xrightarrow{\cong} H^1(A; A^\perp\otimes\operatorname{End} E)$$

maps the extension class of the short exact sequence of A-modules (6) to the Atiyah class of E.

We refer the reader to [12, Lemma 8.2.4] for a related result regarding the Atiyah class of dDG algebras, which correspond to the matched pair case as pointed out in Remark 17.

3. Leibniz$_\infty$ algebras

In this section, we will explore the rich algebraic structures underlying the Atiyah class of a Lie pair. As we will see in the subsequent discussion, the adequate framework is the notion of Leibniz$_\infty$ algebras. Loday's Leibniz$_\infty$ algebras [30] are a natural generalization of Stasheff's L_∞ algebras [28, 27], where the (skew-)symmetry requirement is dropped.

Throughout this section, we implicitly identify objects of $\mathcal{A}$ to complexes in $\mathcal{A}$ concentrated in degree 0. Moreover, we make frequent use of the shifting functor: the shift $V[k]$ of a graded vector space $V = \bigoplus_n V_n$ is determined by the rule $\big(V[k]\big)_n = V_{k+n}$.

We defer most proofs to Section 3.5.

3.1. **Atiyah class as Lie algebra object.**

Recall that a *Lie algebra object* in a monoidal category $\mathcal{C}$ is an object Λ of $\mathcal{C}$ together with a morphism $\lambda \in \operatorname{Hom}_{\mathcal{C}}(\Lambda\otimes\Lambda,\Lambda)$ such that $\lambda\circ\tau = -\lambda$ (skew-symmetry) and

$$\lambda\circ(\operatorname{id}\otimes\lambda) = \lambda\circ(\lambda\otimes\operatorname{id}) + \lambda\circ(\operatorname{id}\otimes\lambda)\circ(\tau\otimes\operatorname{id}) \qquad \text{(Jacobi identity)},$$

where $\tau:\Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is the braiding isomorphism.

Let (L,A) be a Lie pair with quotient $B=L/A$. Note that

$$\operatorname{Hom}_{D^b(\mathcal{A})}(B\otimes B, B[1]) \cong \operatorname{Hom}_{D^b(\mathcal{A})}(B[-1]\otimes B[-1], B[-1]),$$

since a chain map $B\otimes B\to B[1]$ is equivalent to a map $B[-1]\otimes B[-1]\to B[-1]$ modulo a shift. Being an element of

$$\operatorname{Ext}^1_{\mathcal{A}}(B, B^*\otimes B) \cong \operatorname{Ext}^1_{\mathcal{A}}(B\otimes B, B) \cong \operatorname{Hom}_{D^b(\mathcal{A})}(B\otimes B, B[1]) \cong \operatorname{Hom}_{D^b(\mathcal{A})}(B[-1]\otimes B[-1], B[-1]),$$

the Atiyah class α_B of the A-module B defines a "Lie bracket" on $B[-1]$. If, moreover, E is an A-module, its Atiyah class

$$\alpha_E \in \operatorname{Ext}^1_{\mathcal{A}}(E, B^*\otimes E) \cong \operatorname{Ext}^1_{\mathcal{A}}(B\otimes E, E) \cong \operatorname{Hom}_{D^b(\mathcal{A})}(B[-1]\otimes E[-1], E[-1])$$

defines a "representation" on $E[-1]$ of the "Lie algebra" $B[-1]$.

Theorem 45. *Let (L,A) be a Lie pair with quotient $B=L/A$. Then $B[-1]$ is a Lie algebra object in the derived category $D^+(\mathcal{A})$. Moreover, if E is an A-module, then $E[-1]$ is a module object over the Lie algebra object $B[-1]$ in the derived category $D^+(\mathcal{A})$.*

Remark 46. *From the skew-symmetric property of a Lie algebra, it follows that the Atiyah class α_B can indeed be considered as an element in $H^1(A, S^2B^*\otimes B)$, or more precisely, in the image of the map $H^1(A, B^*\otimes\operatorname{End}B)\to H^1(A,S^2B^*\otimes B)$ induced by the A-modules morphism $S^2B^*\otimes B\to B^*\otimes B^*\otimes B(\cong B^*\otimes\operatorname{End}B)$.*

Remark 47. *It is implicitly stated in* [22] *(see also* [39, 37]*) that, if X is a complex manifold, then $T_X[-1]$ is a Lie algebra object in the derived category $D^+(X)$ of bounded below complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. This is simply Theorem 45 in the special case where $L=T_X\otimes\mathbb{C}$ and $A=T_X^{0,1}$.*

3.2. **Jacobi identity up to homotopy.**

Let (L,A) be a Lie pair and E an A-module. The quotient $B=L/A$ is naturally an A-module (see Proposition 4).

Consider the graded vector spaces $V=\bigoplus_{n=0}^\infty\Gamma(\wedge^nA^*\otimes B)$ and $W=\bigoplus_{n=0}^\infty\Gamma(\wedge^nA^*\otimes E)$, and the covariant differentials

$$\partial^A:\Gamma\big(\wedge^\bullet A^*\otimes B\big)\to\Gamma\big(\wedge^{\bullet+1}A^*\otimes B\big)$$
$$\partial^A:\Gamma\big(\wedge^\bullet A^*\otimes E\big)\to\Gamma\big(\wedge^{\bullet+1}A^*\otimes E\big)$$

associated to the A-actions on B and E, respectively. Choosing an L-connection ∇ on L/A extending the A-action, we obtain the bundle maps $R_2 : B \otimes B \to \operatorname{Hom}(A, B)$ and $S_2 : B \otimes E \to \operatorname{Hom}(A, E)$ given by

$$A \ni a \xmapsto{R_2(b_1,b_2)} R_B^{\nabla}(a; b_1)b_2 \in B, \tag{17}$$

$$A \ni a \xmapsto{S_2(b,e)} R_E^{\nabla}(a; b)e \in E, \tag{18}$$

where $R_B^{\nabla} : A \otimes B \to \operatorname{End} B$ and $R_E^{\nabla} : A \otimes B \to \operatorname{End} E$ denote the Atiyah cocycles of B and E.

Theorem 48. *Up to homotopies, the complex $(V[-1], \partial^A)$ is a differential graded Lie algebra and the complex $(W[-1], \partial^A)$ is a differential graded module over it. The Lie algebra bracket*

$$V[-1] \otimes V[-1] \xrightarrow{\lambda} V[-1]$$

and the representation

$$V[-1] \otimes W[-1] \xrightarrow{\mu} W[-1]$$

are given by

$$\lambda\big((\xi_1 \otimes b_1) \otimes (\xi_2 \otimes b_2)\big) = (-1)^{k_2} \xi_1 \wedge \xi_2 \wedge R_2(b_1, b_2)$$

and

$$\mu\big((\xi_1 \otimes b) \otimes (\xi_2 \otimes e)\big) = (-1)^{k_2} \xi_1 \wedge \xi_2 \wedge S_2(b, e),$$

where $\xi_1 \in \Gamma(\wedge^{k_1} A^*)$, $\xi_2 \in \Gamma(\wedge^{k_2} A^*)$, $b_1, b_2, b \in \Gamma(B)$, and $e \in \Gamma(E)$.

Consequently, the cohomology $\bigoplus_{i\geq 1} H^{i-1}(A; E) = H^\bullet(W[-1], \partial^A)$ is a module over the (graded) Lie algebra $\bigoplus_{i\geq 1} H^{i-1}(A; B) = H^\bullet(V[-1], \partial^A)$.

In Section 3.4, we will describe a result which keeps track of higher homotopies.

3.3. **$\text{Leibniz}_\infty[1]$ algebras.** Recall that a graded Leibniz algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$ equipped with a bilinear bracket $V \otimes V \xrightarrow{[-,-]} V$ of degree 0 satisfying the graded Leibniz rule

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

for all homogeneous elements $x, y, z \in V$.

If, moreover, V is endowed with a differential δ of degree 1 satisfying

$$\delta[x, y] = [\delta x, y] + (-1)^{|x|+1}[x, \delta y]$$

for all homogeneous elements $x, y \in V$, then we say that $(V, [-,-], \delta)$ is a differential graded Leibniz algebra. Here $V[n]$ denotes the shifted graded vector space: $(V[n])_k = V_{n+k}$.

Definition 49. *A $\text{Leibniz}_\infty[1]$ algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^{\infty}$ of multilinear maps $\lambda_k : \otimes^k V \to V$ of degree 1 satisfying the identity*

$$\sum_{1\leqslant j\leqslant k\leqslant n} \sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}} \epsilon(\sigma; v_1, \cdots, v_{k-1})(-1)^{|v_{\sigma(1)}|+|v_{\sigma(2)}|+\cdots+|v_{\sigma(k-j)}|}$$
$$\lambda_{n-j+1}\big(v_{\sigma(1)}, \cdots, v_{\sigma(k-j)}, \lambda_j(v_{\sigma(k+1-j)}, \cdots, v_{\sigma(k-1)}, v_k), v_{k+1}, \cdots, v_n\big) = 0 \tag{19}$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \ldots, v_n \in V$. Here $\mathfrak{S}_{k-j}^{j-1}$ denotes the set of $(k-j, j-1)$-shuffles[1]*, and $\epsilon(\sigma; v_1, \cdots, v_{k-1})$ denotes the Koszul sign*[2] *of the permutation σ of the (homogeneous) vectors $v_1, v_2, \ldots, v_{k-1}$.*

Remark 50. *If all λ_k are zero except for λ_1, (V, λ_1) is simply a cochain complex. If $\lambda_k = 0$ $(k \geq 3)$, then $(V[-1], [-,-], d)$ is a graded differential Leibniz algebra, where $[x, y] = (-1)^{|x|}\lambda_2(x, y)$, and $d = \lambda_1$.*

Remark 51. *A graded vector space V is a Leibniz$_\infty[1]$ algebra if and only if the shifted graded vector space $V[-1]$ is a Leibniz$_\infty$ algebra in the sense of Loday [1, 43]. Working with Leibniz$_\infty[1]$ algebras rather than Leibniz$_\infty$ algebras is convenient as all maps in the sequence $(\lambda_k)_{k=1}^n$ have the same degree in this setting.*

Definition 52. *A module over a Leibniz$_\infty[1]$ algebra V is a $\mathbb{Z}$-graded vector space $W = \bigoplus_{n\in\mathbb{Z}} W_n$ together with a sequence $(\mu_k)_{k=1}^\infty$ of multilinear maps*

$$\mu_k : (\otimes^{k-1} V) \otimes W \to W$$

of degree 1 satisfying the identity

$$\begin{aligned}\sum_{1\leqslant j\leqslant k\leqslant n-1} \sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}} &\epsilon(\sigma; v_1, \cdots, v_{k-1})(-1)^{|v_{\sigma(1)}|+|v_{\sigma(2)}|+\cdots+|v_{\sigma(k-j)}|} \\ &\mu_{n-j+1}\big(v_{\sigma(1)}, \cdots, v_{\sigma(k-j)}, \lambda_j(v_{\sigma(k+1-j)}, \cdots, v_{\sigma(k-1)}, v_k), v_{k+1}, \cdots, v_{n-1}, w\big) \\ &+ \sum_{1\leqslant j\leqslant n} \sum_{\sigma\in\mathfrak{S}_{n-j}^{j-1}} \epsilon(\sigma; v_1, \cdots, v_{n-1})(-1)^{|v_{\sigma(1)}|+|v_{\sigma(2)}|+\cdots+|v_{\sigma(n-j)}|} \\ &\qquad \mu_{n-j+1}\big(v_{\sigma(1)}, \cdots, v_{\sigma(n-j)}, \mu_j(v_{\sigma(n+1-j)}, \cdots, v_{\sigma(n-1)}, w)\big) = 0\end{aligned}$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \ldots, v_{n-1} \in V$ and $w \in W$.

Remark 53. *A graded vector space W is a module over a Leibniz$_\infty[1]$ algebra V if and only if $V \oplus W$ is a Leibniz$_\infty[1]$ algebra such that V is a Leibniz$_\infty[1]$ subalgebra [26].*

The proof of the next proposition is a direct verification, which we omit.

Proposition 54. *If $\big(V, (\lambda_k)_{k=1}^\infty\big)$ is a Leibniz$_\infty[1]$ algebra, then (V, λ_1) is a cochain complex and its cohomology $H^\bullet(V)[-1]$ is a graded Leibniz algebra with bracket $H(\lambda_2)$, the image of λ_2 (seen as a chain map) under the cohomology functor. Moreover, if $\big(W, (\mu_k)_{k=1}^\infty\big)$ is a module over $\big(V, (\lambda_k)_{k=1}^\infty\big)$, then (W, μ_1) is a cochain complex and $H(\mu_2)$ is a representation of $H^\bullet(V)[-1]$ on the cohomology $H^\bullet(W)[-1]$ of (W, μ_1).*

3.4. **Main theorem.** Unless we state otherwise, we assume throughout this section that (L, A) is a Lie pair and E is an A-module. The quotient $B = L/A$ is naturally an A-module (see Proposition 4). We use the symbol ∂^A to denote the covariant differential

$$\partial^A : \Gamma\big(\wedge^\bullet A^* \otimes (\otimes^\star B^*) \otimes E\big) \to \Gamma\big(\wedge^{\bullet+1} A^* \otimes (\otimes^\star B^*) \otimes E\big)$$

1. A $(k-j, j-1)$-shuffle is a permutation σ of the set $\{1, 2, \cdots, k-1\}$ such that $\sigma(1) \leqslant \sigma(2) \leqslant \cdots \leqslant \sigma(k-j)$ and $\sigma(k-j+1) \leqslant \sigma(k-j+2) \leqslant \cdots \leqslant \sigma(k-1)$.

2. The Koszul sign of a permutation σ of the (homogeneous) vectors $v_1, v_2, \ldots, v_n$ is determined by the relation $v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(n)} = \epsilon(\sigma; v_1, \cdots, v_n) \cdot v_1 \odot v_2 \odot \cdots \odot v_n$.

associated to the A-action on $(\otimes^\star B^*)\otimes E$. In particular, for any bundle map $\mu : (\wedge^k A)\otimes(\otimes^l B)\to B$, we have

$$\begin{aligned}
&\big(\partial^A\mu\big)(a_0\wedge\cdots\wedge a_k; b_1\otimes\cdots\otimes b_l) = \\
&\qquad\sum_{i=0}^{k}(-1)^i\left\{\nabla_{a_i}\big(\mu(a_{\widehat{i}}; b_1\otimes\cdots\otimes b_l)\big)-\mu\big(a_{\widehat{i}};\nabla_{a_i}(b_1\otimes\cdots\otimes b_l)\big)\right\}\\
&\qquad\qquad+\sum_{i<j}(-1)^{i+j}\mu([a_i,a_j]\wedge a_{\widehat{i,j}}; b_1\otimes\cdots\otimes b_l),
\end{aligned}$$

where $a_{\widehat{i}}$ stands for $a_0\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge a_k$ and $a_{\widehat{i,j}}$ for $a_0\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge\widehat{a_j}\wedge\cdots\wedge a_k$, and $\nabla_{a_i}(b_1\otimes\cdots\otimes b_l)$ for $\sum_{j=1}^{l} b_1\otimes\cdots\otimes\nabla_{a_i}b_j\otimes\cdots\otimes b_l$.

3.4.1. *The operator* ∂^∇. Now choose an extension of the A-action on E to an L-connection ∇ on E, an extension of the A-action on B to an L-connection ∇ on B, and a splitting of the short exact sequence of vector bundles

$$0\longrightarrow A\xrightarrow{\;i\;}L\xrightarrow{\;q\;}B\longrightarrow 0\ , \tag{20}$$

i.e. a pair of maps $j : B\to L$ and $p : L\to A$ such that $q\circ j=\mathrm{id}_B$, $p\circ i=\mathrm{id}_A$ and $i\circ p+j\circ q=\mathrm{id}_L$:

$$\begin{tikzcd} 0 \arrow[r, shift left] & A \arrow[l, shift left] \arrow[r, shift left, "i"] & L \arrow[l, shift left, "p"] \arrow[r, shift left, "q"] & B \arrow[l, shift left, "j"] \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}\ .$$

This splitting determines the map

$$\Gamma(B)\times\Gamma(A)\to\Gamma(A) : (b,a)\mapsto p[j(b),i(a)],$$

which we will denote by Δ since it satisfies the relations

$$\Delta_{fb}a=f\Delta_b a\qquad\text{and}\qquad\Delta_b(fa)=\langle\rho^* df, j(b)\rangle a+f\Delta_b a,$$

for all $f\in C^\infty(M)$, $b\in\Gamma(B)$, and $a\in\Gamma(A)$. In some sense, B "acts" on A.
Identifying sections of $\wedge^\bullet A^*\otimes(\otimes^\star B^*)\otimes E$ with bundle maps $\wedge^\bullet A\otimes(\otimes^\star B)\to E$, we define a differential operator

$$\partial^\nabla : \Gamma(\wedge^\bullet A^*\otimes(\otimes^\star B^*)\otimes E)\to\Gamma(\wedge^\bullet A^*\otimes(\otimes^{\star+1}B^*)\otimes E)$$

by

$$(-1)^k b_0 \lrcorner (\partial^\nabla\omega)=\nabla_{j(b_0)}\omega$$

or, more precisely,

$$\begin{aligned}
(-1)^k\big(\partial^\nabla\omega\big)(a_1,\cdots,a_k;b_0,\cdots,b_l)&=\nabla_{j(b_0)}\big(\omega(a_1,\cdots,a_k;b_1,\cdots,b_l)\big)\\
&-\omega(\Delta_{b_0}a_1,\cdots,a_k;b_1,\cdots,b_l)-\cdots-\omega(a_1,\cdots,\Delta_{b_0}a_k;b_1,\cdots,b_l)\\
&-\omega(a_1,\cdots,a_k;\nabla_{j(b_0)}b_1,\cdots,b_l)-\cdots-\omega(a_1,\cdots,a_k;b_1,\cdots,\nabla_{j(b_0)}b_l),
\end{aligned}$$

where $a_1,\cdots,a_k\in\Gamma(A)$, $b_0,\cdots,b_l\in\Gamma(B)$, and $\omega : \wedge^k A\otimes(\otimes^l B)\to E$.

Note that ∂^∇ depends on the choice of the L-connections extending the A-actions and the splitting $j : B\to L$ of the short exact sequence (20), while ∂^A does not.

The chosen splitting of (20) does also determine three vector bundle maps

$$\alpha : \wedge^2 B\to A,\qquad \beta : \wedge^2 B\to B,\quad\text{and}\quad \Omega : \wedge^2 B\to\operatorname{End} B$$

given by

$$\alpha(b_1,b_2)=p[j(b_1),j(b_2)]$$
$$\beta(b_1,b_2)=\nabla_{j(b_1)}b_2-\nabla_{j(b_2)}b_1-q[j(b_1),j(b_2)]$$

and

$$\Omega(b_1, b_2) = \nabla_{j(b_1)}\nabla_{j(b_2)} - \nabla_{j(b_2)}\nabla_{j(b_1)} - \nabla_{[j(b_1),j(b_2)]}.$$

Proposition 55. *For any $a \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$, we have*

$$R_B^\nabla(a; b_1)b_2 - R_B^\nabla(a; b_2)b_1 = \big(\nabla_a\beta\big)(b_1, b_2)$$

or, equivalently,

$$R_2(b_1, b_2) - R_2(b_2, b_1) = \big(\partial^A\beta\big)(b_1, b_2).$$

Proof. For convenience, we set $\widetilde{b} = j(b)$, $\forall b \in \Gamma(B)$. Hence $[a, \widetilde{b}] = -\Delta_b a + \widetilde{\nabla_a b}$ and

$$[\widetilde{b_1}, \widetilde{b_2}] = \alpha(b_1, b_2) + \widetilde{\nabla_{\widetilde{b_1}} b_2} - \widetilde{\nabla_{\widetilde{b_2}} b_1} - \beta\widetilde{(b_1, b_2)}.$$

A straightforward computation yields the equality

$$\begin{aligned} &q\big([[a, \widetilde{b_1}], \widetilde{b_2}] + [[\widetilde{b_1}, \widetilde{b_2}], a] + [[\widetilde{b_2}, a], \widetilde{b_1}]\big) \\ &\qquad = R_B^\nabla(a; b_2)b_1 - R_B^\nabla(a; b_1)b_2 + \nabla_a\big(\beta(b_1, b_2)\big) - \beta(\nabla_a b_1, b_2) - \beta(b_1, \nabla_a b_2). \end{aligned}$$

The result follows from the Jacobi identity in the Lie algebroid L. ☐

Note that, since R_B^∇ is (by its very definition) independent of the choice of the splitting, Proposition 55 asserts that, unlike β, $\partial^A\beta$ does not depend on the choice of splitting.

3.4.2. *The maps R_n.* Recall the bundle map $R_2 : B \otimes B \to \mathrm{Hom}(A, B)$ associated to the Atiyah cocycle of B given by (17). Since B is an A-module, we can substitute B for E in the definitions of ∂^A and ∂^∇ above and define a sequence of bundle maps

$$R_n : \otimes^n B \to \mathrm{Hom}(A, B)$$

inductively by the relation

$$R_{n+1} = \partial^\nabla R_n, \qquad \text{for } n \geq 2.$$

Hence, we have

$$R_{n+1}(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = R_n\big(\nabla_{j(b_0)}(b_1 \otimes \cdots \otimes b_n)\big) - \nabla_{j(b_0)}\big(R_n(b_1 \otimes \cdots \otimes b_n)\big).$$

Example 56. *Let $L = A \bowtie B$ be a matched pair of Lie algebras. Any bilinear map $\gamma : B \otimes B \to B$ determines an L-connection ∇ on B extending its A-module structure (and conversely): $\nabla_{b_1} b_2 = \gamma(b_1, b_2)$. Taking $\gamma = 0$, the Atiyah cocycle reads $R_B^\nabla(a; b_1)b_2 = \nabla_{\nabla_{b_1} a} b_2$. Hence*

$$R_n(b_1, b_2, b_3, \cdots, b_n) = \nabla_{\nabla_{b_{n-1}}\nabla_{b_{n-2}}\cdots\nabla_{b_1}(-)} b_n.$$

3.4.3. *Leibniz$_\infty$[1] algebra (and modules) arising from a Lie pair.* Consider the sequence of k-ary operations $\lambda_k : \otimes^k V \to V$ ($k \in \mathbb{N}$) on the graded vector space $V = \bigoplus_{n=0}^\infty \Gamma(\wedge^n A^* \otimes B)$ defined by $\lambda_1 = \partial^A$ and, for $k \geq 2$,

$$\lambda_k(\xi_1 \otimes b_1, \cdots, \xi_k \otimes b_k) = (-1)^{|\xi_1|+\cdots+|\xi_k|}\xi_1 \wedge \cdots \wedge \xi_k \wedge R_k(b_1, \cdots, b_k), \tag{21}$$

where $b_1, \ldots, b_k \in \Gamma(B)$ and $\xi_1, \ldots, \xi_k$ are homogeneous elements of $\Gamma(\wedge^\bullet A^*)$.

Theorem 57. *When endowed with the sequence of multibrackets $(\lambda_k)_{k\in\mathbb{N}}$ defined above, the graded vector space $V = \bigoplus_{n=0}^\infty \Gamma(\wedge^n A^* \otimes B)$ becomes a Leibniz$_\infty$[1] algebra.*

Similarly, we can introduce the bundle map $S_2 : B \otimes E \to \mathrm{Hom}(A, E)$ given by

$$A \ni a \xmapsto{S_2(b;e)} R_E^{\nabla}(a;b) \cdot e \in E,$$

where $R_E^{\nabla} : A \otimes B \to \mathrm{End}\, E$ denotes the Atiyah cocycle of the A-module E, and then define a sequence of bundle maps

$$S_n : (\otimes^{n-1} B) \otimes E \to \mathrm{Hom}(A, E)$$

inductively by the relation

$$S_{n+1} = \partial^{\nabla} S_n, \qquad \text{for } n \geq 2.$$

This leads to the graded vector space $W = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes E)$ and the sequence of k-ary brackets $\mu_k : (\otimes^{k-1} V) \otimes W \to W$ $(k \in \mathbb{N})$ defined by $\mu_1 = \partial^A$ and, for $k \geq 2$,

$$\mu_k(\xi_1 \otimes b_1, \cdots, \xi_{k-1} \otimes b_{k-1}; \xi_k \otimes e) = (-1)^{|\xi_1|+\cdots+|\xi_k|} \xi_1 \wedge \cdots \wedge \xi_k \wedge S_k(b_1, \cdots, b_{k-1}; e),$$

where $b_1, \ldots, b_{k-1} \in \Gamma(B)$, $e \in \Gamma(E)$, and $\xi_1, \ldots, \xi_k$ are homogeneous elements of $\Gamma(\wedge^\bullet A^*)$.

Theorem 58. *When endowed with the sequence of multibrackets $(\mu_k)_{k\in\mathbb{N}}$ defined above, the graded vector space $W = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes E)$ becomes a Leibniz$_\infty[1]$ module over the Leibniz$_\infty[1]$ algebra $\big(V, (\lambda_k)_{k\in\mathbb{N}}\big)$.*

Example 59. *A Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a matched pair of Lie algebras. Therefore, it induces two Lie pairs: $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g})$ and $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}^*)$. It follows from Example 56 and Theorem 57 that both $\bigoplus_{n\geq 0} \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^*$ and $\bigoplus_{n\geq 0} \wedge^n \mathfrak{g} \otimes \mathfrak{g}$ are Leibniz$_\infty[1]$ algebras.*

Let A be a Lie algebroid over a manifold M. By an A-algebra, we mean a bundle (of finite or infinite rank) of associative algebras $\mathcal{C}$ over M, which is an A-module, and on which $\Gamma(A)$ acts by derivations. For a commutative A-algebra $\mathcal{C}$, the sequence of maps $(\lambda_k)_{k\in\mathbb{N}}$ extends in a natural way to the graded space $\bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes B \otimes \mathcal{C})$. Similarly, the sequence of maps $(\mu_k)_{k\in\mathbb{N}}$ extends to the graded space $\bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes E \otimes \mathcal{C})$.

Theorem 60. *Let (L, A) be a Lie pair with quotient B, and let $\mathcal{C}$ be a commutative A-algebra. When endowed with the sequence of multibrackets $(\lambda_k)_{k\in\mathbb{N}}$, the graded vector space $\Gamma(\wedge^\bullet A^* \otimes B \otimes \mathcal{C})$ becomes a Leibniz$_\infty[1]$ algebra. Moreover, if E is an A-module, the graded vector space $\Gamma(\wedge^\bullet A^* \otimes E \otimes \mathcal{C})[-1]$ becomes a Leibniz$_\infty[1]$ module over the Leibniz$_\infty[1]$ algebra $\Gamma(\wedge^\bullet A^* \otimes B \otimes \mathcal{C})[-1]$.*

As an immediate consequence, we have the following

Corollary 61. *Under the same hypothesis as in Theorem 60, $\bigoplus_{i\geq 1} H^{i-1}(A, B \otimes \mathcal{C})$ is a graded Lie algebra and $\bigoplus_{i\geq 1} H^{i-1}(A, E \otimes \mathcal{C})$ a module over it.*

Example 62. *Let $\mathfrak{g}$ be a Lie subalgebra of a Lie algebra $\mathfrak{d}$ as in Example 21. Assume that $\mathcal{C}$ is a commutative $\mathfrak{g}$-algebra.*
Every linear map $\boldsymbol{L} : \mathfrak{d} \to \mathrm{End}(\mathfrak{d}/\mathfrak{g})$ that extends the $\mathfrak{g}$-module structure $\mathfrak{g} \to \mathrm{End}(\mathfrak{d}/\mathfrak{g})$ induces a 2-ary bracket on $\wedge^{\bullet-1}\mathfrak{g}^* \otimes \mathfrak{d}/\mathfrak{g} \otimes \mathcal{C}$:

$$[\xi_1 \otimes b_1 \otimes c_1, \xi_2 \otimes b_2 \otimes c_2] = (-1)^{|\xi_2|} \xi_1 \wedge \xi_2 \otimes \big(\partial^{\mathfrak{g}} \boldsymbol{L}\big)(-; b_1) \cdot b_2 \otimes c_1 c_2, \tag{22}$$

which in turn induces a (graded) Lie algebra bracket on the Chevalley-Eilenberg cohomology $\bigoplus H^{\bullet-1}(\mathfrak{g}; \mathfrak{d}/\mathfrak{g} \otimes \mathcal{C})$. Here $\xi_i \otimes b_i \otimes c_i$ $(i = 1, 2)$ are cocycles with $\xi_1, \xi_2 \in \wedge^\bullet \mathfrak{g}^*$, $b_1, b_2 \in \mathfrak{d}/\mathfrak{g}$ and $c_1, c_2 \in \mathcal{C}$.
Moreover, if E is a $\mathfrak{g}$-module, every linear map $\boldsymbol{M} : \mathfrak{d} \to \mathrm{End}\, E$ that extends the $\mathfrak{g}$-module structure $\mathfrak{g} \to \mathrm{End}\, E$ gives rise to a bilinear map

$$(\xi_1 \otimes b \otimes c_1) \rhd (\xi_2 \otimes e \otimes c_2) = (-1)^{|\xi_2|} \xi_1 \wedge \xi_2 \otimes \big(\partial^{\mathfrak{g}} \boldsymbol{M}\big)(-; b) \cdot e \otimes c_1 c_2,$$

which induces a representation on $\bigoplus H^{\bullet-1}(\mathfrak{g}; E\otimes\mathcal{C})$ *of the graded Lie algebra* $\bigoplus H^{\bullet-1}(\mathfrak{g}; \mathfrak{d}/\mathfrak{g}\otimes\mathcal{C})$*. Here* $\xi_1 \otimes b \otimes c_1$ *and* $\xi_2 \otimes e \otimes c_2$ *are cocycles with* $\xi_1, \xi_2 \in \wedge^\bullet\mathfrak{g}^*$*,* $b \in \mathfrak{d}/\mathfrak{g}$*,* $e \in E$ *and* $c_1, c_2 \in \mathcal{C}$*.*

Take a complement $\mathfrak{h}$ *of* $\mathfrak{g}$ *in* $\mathfrak{d}$ *so that we can write* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$*. Then* $\mathfrak{d}/\mathfrak{g}$ *can be identified with* $\mathfrak{h}$*, on which the* $\mathfrak{h}$*-action is given by* $a \cdot h = \mathrm{pr}_\mathfrak{h}[a, h]$*. Take* $\boldsymbol{L} : \mathfrak{d} \to \mathrm{End}\,\mathfrak{h}$ *to be the trivial extension of the* $\mathfrak{g}$*-module structure* $\mathfrak{g} \to \mathrm{End}\,\mathfrak{h}$*, i.e. set* $\boldsymbol{L}|_\mathfrak{h} = 0$*. Then the 2-ary bracket in Equation (22) is given by*

$$[f \otimes c_1,\ g \otimes c_2] = [f, g] \otimes c_1c_2,$$

where $f \in \wedge^p\mathfrak{g} \to \mathfrak{h}$*,* $g \in \wedge^q\mathfrak{g} \to \mathfrak{h}$*,* $c_1, c_2 \in \mathcal{C}$*, and* $[f, g] \in \wedge^{p+q+1}\mathfrak{g} \to \mathfrak{h}$ *is given by*

$$[f,g](a_0,a_1,\cdots,a_{p+q}) = \\ -\sum_{\sigma\in\mathfrak{S}_{1,p,q}} \mathrm{sgn}(\sigma)\,\mathrm{pr}_\mathfrak{h}[\mathrm{pr}_\mathfrak{g}[a_{\sigma(0)}, f(a_{\sigma(1)},\cdots,a_{\sigma(p)})], g(a_{\sigma(p+1)},\cdots,a_{\sigma(p+q)})],$$

where $\mathfrak{S}_{1,p,q}$ *is the set of all permutations* σ *of* $\{0, 1, \cdots, p+q\}$ *satisfying* $\sigma(1) < \cdots < \sigma(p)$ *and* $\sigma(p+1) < \cdots < \sigma(p+q)$*.*

Remark 63. *It is natural to ask how the* Leibniz$_\infty[1]$ *algebra structure obtained in Theorem 57 and the* Leibniz$_\infty[1]$ *module structure in Theorem 58 depend on the choice of connections and the splitting data. This question will be investigated somewhere else.*

3.4.4. L_∞ *rather than Leibniz*$_\infty$*.*

Theorem 64. *Let* (A, B) *be a matched pair of Lie algebroids with direct sum* $L = A \bowtie B$*. Assume there exists a flat torsion free* B*-connection on* B*. Then the maps* R_n *defined above are totally symmetric, the multibrackets* $\lambda_k : \otimes^k V \to V$ $(k \in \mathbb{N})$ *defined earlier on the graded vector space* $V = \bigoplus_{n=0}^{\infty} \Gamma(\wedge^n A^* \otimes B)$ *are graded symmetric, and* $V[-1]$ *is actually an* L_∞ *algebra.*

The following example is due to Camille Laurent-Gengoux.

Example 65. *The general Lie algebra* $\mathfrak{gl}_n(\mathbb{C})$ *decomposes as the direct sum of the unitary Lie algebra* $\mathfrak{u}_n$ *and the Lie algebra* $\mathfrak{t}_n$ *of upper triangular matrices with real diagonal coefficients. Both* $\mathfrak{u}_n$ *and* $\mathfrak{t}_n$ *are isotropic with respect to the natural nondegenerate ad-invariant inner product* $X \otimes Y \mapsto \mathrm{im}\big(\mathrm{tr}(XY)\big)$ *on* $\mathfrak{gl}_n(\mathbb{C})$*. Hence* $(\mathfrak{u}_n, \mathfrak{t}_n)$ *is a matched pair of Lie algebras as well as a Lie bialgebra. Matrix multiplication being associative, setting* $\nabla_X Y = XY$ *for any* $X, Y \in \mathfrak{t}_n$ *defines a flat torsion free* $\mathfrak{t}_n$*-connection on* $\mathfrak{t}_n$*. It follows from Theorem 64 that* $\Gamma(\wedge^\bullet\mathfrak{u}_n^* \otimes \mathfrak{t}_n)[-1] \cong \Gamma(\wedge^\bullet\mathfrak{t}_n \otimes \mathfrak{t}_n)[-1]$ *is an* L_∞ *algebra.*

Example 66 (Kapranov)**.** *Suppose* X *is a Kähler manifold. The complexification* $\nabla^\mathbb{C}$ *of its Levi-Civita connection is a* $T_X \otimes \mathbb{C}$*-connection on* $T_X \otimes \mathbb{C}$*. Set* $A = T_X^{0,1}$ *and* $B = T_X^{1,0}$*. Then* (A, B) *is a matched pair of Lie algebroids, whose direct sum* $A \bowtie B$ *is isomorphic, as a Lie algebroid, to* $T_X \otimes \mathbb{C}$*. It is easy to see that* $\nabla^\mathbb{C}$ *induces a flat torsion free* B*-connection on* B*. In this context, the tensors* $R_n \in \Omega^{0,1}\big(\mathrm{Hom}(\otimes^n T, T)\big)$ *(where* T *stands for* $T_X^{1,0}$*) are the curvature* $R_2 \in \Omega^{1,1}(\mathrm{End}\,T) \cong \Omega^{0,2}\big(\mathrm{Hom}(T \otimes T, T)\big)$ *and its higher covariant derivatives:* $R_{i+1} = \partial^\nabla R_i$*. Applying Theorem 64, we recover a result of Kapranov [22]:*

Corollary 67 (Kapranov)**.** *The shifted Dolbeault complex* $\Omega^{0,\bullet-1}(T)$ *of a Kähler manifold is an* L_∞ *algebra. The* n*-th multibracket*

$$\lambda_n : \Omega^{0,j_1}(T) \otimes \cdots \otimes \Omega^{0,j_n}(T) \to \Omega^{0,j_1+\cdots+j_n+1}(T)$$

is the composition of the wedge product

$$\Omega^{0,j_1}(T) \otimes \cdots \otimes \Omega^{0,j_n}(T) \to \Omega^{0,j_1+\cdots+j_n}(\otimes^n T)$$

with the map

$$\Omega^{0,j_1+\cdots+j_n}(\otimes^n T) \to \Omega^{0,j_1+\cdots+j_n+1}(T)$$

associated to $R_n \in \Omega^{0,1}\big(\operatorname{Hom}(\otimes^n T, T)\big)$ *in the obvious way.*

3.5. **Proofs.** This section is devoted to the proofs of the theorems claimed in Section 3. For convenience, we set $\widetilde{b} = j(b)$, $\forall b \in \Gamma(B)$.

3.5.1. *Atiyah class as Lie algebra object.*

Lemma 68. *For any* $a_1, a_2 \in \Gamma(A)$ *and* $b \in \Gamma(B)$*, we have*

$$[\Delta_b a_1, a_2] + [a_1, \Delta_b a_2] - \Delta_b[a_1, a_2] = \Delta_{\nabla_{a_1} b} a_2 - \Delta_{\nabla_{a_2} b} a_1.$$

Proof. We have

$$\begin{aligned}
& p\big([[\widetilde{b}, a_1], a_2] + [[a_1, a_2], \widetilde{b}] + [[a_2, \widetilde{b}], a_1]\big) \\
=& p[p[\widetilde{b}, a_1] + q\widetilde{[\widetilde{b}, a_1]}, a_2] - p[\widetilde{b}, [a_1, a_2]] + p[p[a_2, \widetilde{b}] + q\widetilde{[a_2, \widetilde{b}]}, a_1] \\
=& [p[\widetilde{b}, a_1], a_2] - p[q\widetilde{[a_1, \widetilde{b}]}, a_2] - p[\widetilde{b}, [a_1, a_2]] + [a_1, p[\widetilde{b}, a_2]] + p[q\widetilde{[a_2, \widetilde{b}]}, a_1] \\
=& [\Delta_b a_1, a_2] - \Delta_{\nabla_{a_1} b} a_2 - \Delta_b[a_1, a_2] + [a_1, \Delta_b a_2] + \Delta_{\nabla_{a_2} b} a_1.
\end{aligned}$$

The result follows from the Jacobi identity in the Lie algebroid L. $\square$

Note that a bundle map $\omega : (\wedge^k A) \otimes (\otimes^l B) \to E$ determines a bundle map $\overleftarrow{\omega} : \wedge^k A \to (\otimes^l B^*) \otimes E$ and vice versa.

Lemma 69. *For any bundle map* $\omega : (\wedge^k A) \otimes (\otimes^l B) \to E$*, any* $a_0, \ldots, a_k \in \Gamma(A)$ *and any* $b_0, \ldots, b_l \in \Gamma(B)$*, we have*

$$\begin{aligned}
&(-1)^k\big(\partial^A \partial^\nabla \omega + \partial^\nabla \partial^A \omega\big)(a_0, \cdots, a_k; b_0, \ldots, b_l) \\
&= \sum_{i=0}^{k} (-1)^i \left\langle \nabla_{a_i} \nabla_{\widetilde{b_0}} \big(\overleftarrow{\omega}(\widehat{a_i})\big) - \nabla_{\widetilde{b_0}} \nabla_{a_i} \big(\overleftarrow{\omega}(\widehat{a_i})\big) - \nabla_{[a_i, \widetilde{b_0}]} \big(\overleftarrow{\omega}(\widehat{a_i})\big) \middle| b_1 \otimes \cdots \otimes b_l \right\rangle \\
&= \sum_{i=0}^{k} (-1)^i \left\{ R_E^\nabla(a_i; b_0) \cdot \omega(\widehat{a_i}; \widehat{b_0}) - \sum_{j=1}^{l} \omega\big(\widehat{a_i}; b_1, \cdots, R_B^\nabla(a_i; b_0) \cdot b_j, \cdots, b_l\big) \right\},
\end{aligned}$$

where $\widehat{a_i}$ *stands for* $a_0 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_k$ *and* $\widehat{b_0}$ *for* $b_1 \otimes \cdots \otimes b_l$.

Sketch of proof. The first equality follows from a cumbersome computation at the last step of which use is made of Lemma 68. The second equality is immediate. $\square$

Given $\mu \in \Gamma((\wedge^{k_1} A^*) \otimes (\otimes^{l_1} B^*) \otimes B)$, $\nu \in \Gamma((\wedge^{k_2} A^*) \otimes (\otimes^{l_2} B^*) \otimes B)$, and arbitrary sections $b_1, \ldots, b_{l_1}, b'_1, \ldots, b'_{l_1}$ of B, we can consider the bundle map

$$\big\lceil \mu\big(b_1, \cdots, b_{i-1}, \nu(b'_1, \cdots, b'_{l_2}), b_{i+1}, \cdots, b_{l_1}\big) \big\rfloor : \wedge^{k_1+k_2} A \to B,$$

which maps $a_1 \wedge \cdots \wedge a_{k_1+k_2}$ to

$$\begin{aligned}
\sum_{\sigma \in \mathfrak{S}_{k_1}^{k_2}} \operatorname{sgn}(\sigma) \mu\big(a_{\sigma(1)}, \cdots, a_{\sigma(k_1)}; b_1, \cdots, b_{i-1}, \\
\nu(a_{\sigma(k_1+1)}, \cdots, a_{\sigma(k_1+k_2)}; b'_1, \cdots, b'_{l_2}), b_{i+1}, \cdots, b_{l_1}\big).
\end{aligned}$$

In particular, if $\mu = \alpha_1 \otimes \beta_1 \otimes u$ and $\nu = \alpha_2 \otimes \beta_2 \otimes v$ with $\alpha_1 \in \Gamma(\wedge^{k_1} A^*)$, $\alpha_2 \in \Gamma(\wedge^{k_2} A^*)$, $\beta_1 \in \Gamma(\otimes^{l_1} B^*)$, $\beta_2 \in \Gamma(\otimes^{l_2} B^*)$, and $u, v \in \Gamma(B)$, then

$$\lceil \mu\big(b_1, \cdots, b_{i-1}, \nu(b'_1, \cdots, b'_{l_2}), b_{i+1}, \cdots, b_{l_1}\big) \rfloor = \\ \beta_1(b_1, \cdots, b_{i-1}, v, b_{i+1}, \cdots, b_{l_1})\beta_2(b'_1, \cdots, b'_{l_2}) \cdot (\alpha_1 \wedge \alpha_2) \otimes u.$$

Corollary 70. *For any $n \geq 2$ and $b_0, \ldots, b_n \in \Gamma(B)$, we have*

$$-\big((\partial^A \partial^\nabla + \partial^\nabla \partial^A) R_n\big)(b_0, \cdots, b_n) = \lceil R_2\big(b_0, R_n(b_1, \cdots, b_n)\big) \rfloor \\ + \sum_{j=1}^{n} \lceil R_n\big(b_1, \cdots, R_2(b_0, b_j), \cdots, b_n\big) \rfloor .$$

Proof. Apply Lemma 69 to $\omega = R_n$. □

Corollary 71. *For any $b_0, b_1, b_2 \in \Gamma(B)$, we have*

$$-\big(\partial^A R_3\big)(b_0, b_1, b_2) = \lceil R_2\big(b_0, R_2(b_1, b_2)\big) \rfloor + \lceil R_2\big(R_2(b_0, b_1), b_2\big) \rfloor \\ + \lceil R_2\big(b_1, R_2(b_0, b_2)\big) \rfloor .$$

Proof. Since $\partial^A R_2 = 0$ and $\partial^\nabla R_2 = R_3$, taking $n = 2$ in Corollary 70 yields the result. □

Sketch of proof of Theorem 45. The interchange isomorphism $\tau : B[-1] \otimes B[-1] \to B[-1] \otimes B[-1]$ is the image in $D(\mathcal{A})$ of the chain map $\tau : B[-1] \otimes B[-1] \to B[-1] \otimes B[-1]$ given by $\tau(b_1 \otimes b_2) = -b_2 \otimes b_1$, $\forall b_1, b_2 \in B$ — the negative sign is due to $B[-1]$ being a complex concentrated in degree 1, see Equation (12). Recall that R_2 is a cocycle (w.r.t. ∂^A). Its cohomology class α_B, the Atiyah class of B, can be seen as an element of $\mathrm{Hom}_{D^+(\mathcal{A})}(B[-1] \otimes B[-1], B[-1])$. Proposition 55 implies the equality $\alpha_B \circ \tau = -\alpha_B$ in $\mathrm{Hom}_{D^+(\mathcal{A})}(B[-1] \otimes B[-1], B[-1])$. Corollary 71 implies that the Jacobi identity $\alpha_B \circ (\mathrm{id} \otimes \alpha_B) = \alpha_B \circ (\alpha_B \otimes \mathrm{id}) + \alpha_B \circ (\mathrm{id} \otimes \alpha_B) \circ (\tau \otimes \mathrm{id})$ holds in $D^+(\mathcal{A})$. Indeed, each of the terms $\lceil \cdots \rfloor$ can be interpreted as a Yoneda product, a composition of morphisms in the derived category. □

3.5.2. *Jacobi identity up to homotopy.* Consider the cochain complex $(V[-1], \partial^A)$, where the graded vector space $V = \bigoplus_{k=0}^{\infty} V_k$ is given by $V_k = \Gamma(\wedge^k A^* \otimes B)$ so that, if $\xi \in \Gamma(\wedge^k A^*)$ and $b \in \Gamma(B)$, then $\xi \otimes b \in \big(V[-1]\big)_{k+1}$.

Lemma 72. *The graded linear map $\lambda : V[-1] \otimes V[-1] \to V[-1]$ given by*

$$\lambda(v_1 \otimes v_2) = (-1)^{k_2} \xi_1 \wedge \xi_2 \wedge R_2(b_1, b_2)$$

for any $v_1 = \xi_1 \otimes b_1 \in (V[-1])_{k_1+1}$ and $v_2 = \xi_2 \otimes b_2 \in (V[-1])_{k_2+1}$ is a chain map.

Proof. A straightforward computation yields

$$\big(\partial^A \circ \lambda - \lambda \circ \partial^A\big)(v_1 \otimes v_2) = (-1)^{k_1} \xi_1 \wedge \xi_2 \wedge \big(\partial^A R_2\big)(b_1, b_2).$$

The result follows from $\partial^A R_2 = 0$ (see Theorem 16 and the definition (17) of R_2). □

Now, consider the interchange isomorphism $\tau : V[-1] \otimes V[-1] \to V[-1] \otimes V[-1]$ given by $\tau(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$.

Lemma 73. *The chain map* λ *is skew-symmetric up to a homotopy:*

$$\lambda + \lambda \circ \tau = \partial^A \circ \Theta + \Theta \circ \partial_A,$$

where the graded map $\Theta : V[-1] \otimes V[-1] \to V[-2]$ *is given by*

$$\Theta(v_1 \otimes v_2) = (-1)^{k_1} \xi_1 \wedge \xi_2 \otimes \beta(b_1, b_2)$$

for any $v_1 = \xi_1 \otimes b_1 \in (V[-1])_{k_1+1}$ *and* $v_2 = \xi_2 \otimes b_2 \in (V[-1])_{k_2+1}$*.*

Proof. Straightforward computations yield

$$\big(\lambda + \lambda \circ \tau\big)(v_1 \otimes v_2) = (-1)^{k_2} \xi_1 \wedge \xi_2 \wedge \{R_2(b_1, b_2) - R_2(b_2, b_1)\}$$

and

$$\big(\partial^A \circ \Theta + \Theta \circ \partial_A\big)(v_1 \otimes v_2) = (-1)^{k_2} \xi_1 \wedge \xi_2 \wedge \big\{\big(\partial^A \beta\big)(b_1, b_2)\big\}$$

The result follows from Proposition 55. □

Lemma 74. *The chain map* λ *satisfies the Jacobi identity up to a homotopy:*

$$-\lambda \circ (\mathrm{id} \otimes \lambda) + \lambda \circ (\lambda \otimes \mathrm{id}) + \lambda \circ (\mathrm{id} \otimes \lambda) \circ (\tau \otimes \mathrm{id}) = \partial^A \circ \Xi + \Xi \circ \partial_A,$$

where the graded map $\Xi : V[-1] \otimes V[-1] \otimes V[-1] \to V[-2]$ *is given by*

$$\Xi(v_0 \otimes v_1 \otimes v_2) = (-1)^{k_0+k_2} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge R_3(b_0, b_1, b_2)$$

for any $v_i = \xi_i \otimes b_i \in (V[-1])_{k_i+1}$ *with* $i \in \{0, 1, 2\}$*.*

Proof. Straightforward computations yield

$$\begin{aligned}
\big(\lambda \circ (\mathrm{id} \otimes \lambda)\big)(v_0 \otimes v_1 \otimes v_2) &= (-1)^{k_1} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \lceil R_2(b_0, R_2(b_1, b_2)) \rfloor, \\
\big(\lambda \circ (\lambda \otimes \mathrm{id})\big)(v_0 \otimes v_1 \otimes v_2) &= -(-1)^{k_1} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \lceil R_2(R_2(b_0, b_1), b_2) \rfloor, \\
\big(\lambda \circ (\mathrm{id} \otimes \lambda) \circ (\tau \otimes \mathrm{id})\big)(v_0 \otimes v_1 \otimes v_2) &= -(-1)^{k_1} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \lceil R_2(b_1, R_2(b_0, b_2)) \rfloor,
\end{aligned}$$

and

$$\big(\partial^A \circ \Theta + \Theta \circ \partial_A\big)(v_0 \otimes v_1 \otimes v_2) = (-1)^{k_1} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \big\{\big(\partial^A R_3\big)(b_0, b_1, b_2)\big\}.$$

The result follows from Corollary 71. □

Theorem 48 immediately follows from Lemmas 72, 73, and 74.

Note that Theorem 48 could also be seen as a corollary of Theorems 57 and 58.

3.5.3. *Leibniz*$_\infty$[1] *algebra (and modules) arising from a Lie pair.*

Lemma 75. *For any* $n \geq 3$ *and* $b_1, \ldots, b_n$ *are arbitrary sections of* B*, we have*

$$- \big(\partial^A R_n\big)(b_1, \ldots, b_n) = \sum_{\substack{i+j=n+1 \\ i \geq 2 \\ j \geq 2}} \sum_{k=j}^{n} \sum_{\sigma \in \mathfrak{S}_{k-j}^{j-1}} \lceil R_i\big(b_{\sigma(1)}, \cdots, b_{\sigma(k-j)}, R_j(b_{\sigma(k+1-j)}, \cdots, b_{\sigma(k-1)}, b_k), b_{k+1}, \cdots, b_n\big) \rfloor.$$

Proof. We reason by induction. The formula holds for $n = 3$ by Corollary 71. Assuming the formula holds for $n = N$, we get

$$\big(\partial^\nabla \partial^A R_N\big)(b_0, \ldots, b_N) = \big(\nabla_{b_0}(\partial^A R_N)\big)(b_1, \ldots, b_N) = \sum_{\substack{i+j=N \\ i\geq 2 \\ j\geq 2}} \sum_{k=j}^{N} \sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}}$$

$$\Big\{ \big\lceil R_{i+1}\big(b_0, b_{\sigma(1)}, \cdots, b_{\sigma(k-j)}, R_j(b_{\sigma(k+1-j)}, \cdots, b_{\sigma(k-1)}, b_k), b_{k+1}, \cdots, b_n\big) \big\rfloor$$
$$+ \big\lceil R_i\big(b_{\sigma(1)}, \cdots, b_{\sigma(k-j)}, R_{j+1}(b_0, b_{\sigma(k+1-j)}, \cdots, b_{\sigma(k-1)}, b_k), b_{k+1}, \cdots, b_n\big) \big\rfloor \Big\}.$$

Observing that

$$\partial^A R_{N+1} = (\partial^A \partial^\nabla + \partial^\nabla \partial^A) R_N + \partial^\nabla \partial^A R_N$$

and recalling Corollary 70, it is easy to check that the desired formula holds for $n = N+1$ as well. $\square$

Lemma 76. *For any bundle map $\omega : (\wedge^k A) \otimes (\otimes^l B) \to B$ and any $b_1, \ldots, b_l \in \Gamma(B)$, we have*

$$\partial^A\big(\overleftarrow{\omega}(b_1, \ldots, b_l)\big) - \big(\overleftarrow{\partial^A \omega}\big)(b_1, \ldots, b_l) = (-1)^k \sum_{j=0}^{l} \big\lceil \omega(b_1, \cdots, \partial^A b_j, \cdots, b_l) \big\rfloor .$$

Proof. For any $a_0, \ldots, a_k \in \Gamma(A)$, we have

$$\begin{aligned}
&\Big\langle \partial^A\big(\overleftarrow{\omega}(b_1, \ldots, b_l)\big) - \big(\overleftarrow{\partial^A \omega}\big)(b_1, \ldots, b_l) \Big| a_0 \wedge \cdots \wedge a_k \Big\rangle \\
&= \sum_{j=0}^{l} \sum_{i=0}^{k} (-1)^i \omega(a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k; b_1, \cdots, \nabla_{a_i} b_j, \cdots, b_l) \\
&= (-1)^k \sum_{j=0}^{l} \sum_{\sigma\in\mathfrak{S}_k^1} \operatorname{sgn}(\sigma) \omega(a_{\sigma(0)}, \cdots, a_{\sigma(k-1)}; b_1, \cdots, \nabla_{a_{\sigma(k)}} b_j, \cdots, b_l) \\
&= (-1)^k \sum_{j=0}^{l} \big\langle \big\lceil \omega(b_1, \cdots, \partial^A b_j, \cdots, b_l) \big\rfloor \big| a_0 \wedge \cdots \wedge a_k \big\rangle .
\end{aligned}$$
$\square$

Proof of Theorem 57. We only need to check that the generalized Leibniz identity (19) holds. Since $\lambda_1 = \partial^A$ and $(\partial^A)^2 = 0$, Equation (19) is obviously true for $n = 1$. Let $n \geq 2$ and $v_i = \xi_i \otimes b_i \in \Gamma(\wedge^{p_i} A^* \otimes B)$ for all $i \in \{1, \ldots, n\}$. The l.h.s. of (19) is

$$\sum_{1\leqslant j\leqslant k\leqslant n} \sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}} \epsilon(\sigma; v_1, \cdots, v_{k-1}) (-1)^{|v_{\sigma(1)}|+|v_{\sigma(2)}|+\cdots+|v_{\sigma(k-j)}|}$$
$$\lambda_{n-j+1}\big(v_{\sigma(1)}, \cdots, v_{\sigma(k-j)}, \lambda_j(v_{\sigma(k+1-j)}, \cdots, v_{\sigma(k-1)}, v_k), v_{k+1}, \cdots, v_n\big).$$

Separating the terms involving λ_1 (aka ∂^A) from the others, it can be rewritten as

$$\partial^A\big(\lambda_n(v_1,\cdots,v_n)\big)+\sum_{\substack{i+j=n+1\\ i\geq 2\\ j\geq 2}}\sum_{k=j}^{n}\sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}}\epsilon(\sigma;\xi_1,\cdots,\xi_{k-1})(-1)^{p_{\sigma(1)}+\cdots+p_{\sigma(k-j)}}$$
$$\lambda_i\big(v_{\sigma(1)},\cdots,v_{\sigma(k-j)},\lambda_j(v_{\sigma(k+1-j)},\cdots,v_{\sigma(k-1)},v_k),v_{k+1},\cdots,v_n\big)$$
$$+\sum_{k=1}^{n}(-1)^{p_1+\cdots+p_{k-1}}\lambda_n\big(v_1,\cdots,v_{k-1},(\partial^A\xi_k)\otimes b_k+(-1)^{p_k}\xi_k\otimes(\partial^A b_k),v_{k+1},\cdots,v_n\big)$$

and then, using the definition (21) of each λ_k in terms of the corresponding R_k, as

$$\partial^A\big((-1)^{p_1+\cdots+p_n}\xi_1\wedge\cdots\wedge\xi_n\wedge R_n(b_1,\cdots,b_n)\big)$$
$$+\sum_{\substack{i+j=n+1\\ i\geq 2\\ j\geq 2}}\sum_{k=j}^{n}\sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}}\epsilon(\sigma;\xi_1,\cdots,\xi_{k-1})\xi_{\sigma(1)}\wedge\cdots\wedge\xi_{\sigma(k-1)}\wedge\xi_k\wedge\cdots\wedge\xi_n\wedge$$
$$\big\lceil R_i\big(b_{\sigma(1)},\cdots,b_{\sigma(k-j)},R_j(b_{\sigma(k+1-j)},\cdots,b_{\sigma(k-1)},b_k),b_{k+1},\cdots,b_n\big)\big\rfloor$$
$$+\sum_{k=1}^{n}(-1)^{1+p_k+p_{k+1}+\cdots+p_n}\xi_1\wedge\cdots\wedge\xi_{k-1}\wedge\partial^A\xi_k\wedge\xi_{k+1}\wedge\cdots\wedge\xi_n\wedge R_n(b_1,\cdots,b_n)$$
$$+\sum_{k=1}^{n}\xi_1\wedge\cdots\wedge\xi_n\wedge\big\lceil R_n(b_1,\cdots,\partial^A b_k,\cdots,b_n)\big\rfloor,$$

which simplifies to

$$\xi_1\wedge\cdots\wedge\xi_n\wedge\Big\{\partial^A\big(R_n(b_1,\cdots,b_n)\big)+\sum_{\substack{i+j=n+1\\ i\geq 2\\ j\geq 2}}\sum_{k=j}^{n}\sum_{\sigma\in\mathfrak{S}_{k-j}^{j-1}}$$
$$\big\lceil R_i\big(b_{\sigma(1)},\cdots,b_{\sigma(k-j)},R_j(b_{\sigma(k+1-j)},\cdots,b_{\sigma(k-1)},b_k),b_{k+1},\cdots,b_n\big)\big\rfloor$$
$$+\sum_{k=1}^{n}\big\lceil R_n(b_1,\cdots,\partial^A b_k,\cdots,b_n)\big\rfloor\Big\}.$$

The result now follows from Lemmas 75 and 76. $\square$

The proof of Theorem 58, Theorem 60 and Corollary 61 goes along the same line *mutatis mutandis.*

3.5.4. *L_∞ rather than Leibniz$_\infty$.*

Lemma 77. *For any $b_0,b_1,b_2\in\Gamma(B)$, we have*

$$R_3(b_0,b_1,b_2)-R_3(b_1,b_0,b_2)=R_2\big(\beta(b_0,b_1),b_2\big)-\big(\partial^A\Omega\big)(b_0,b_1)\cdot b_2.$$

Proof. The differential Bianchi identity $d^\nabla R^\nabla=0$ holds for the curvature $R^\nabla:\wedge^2L\to\operatorname{End}B$ of the L-connection ∇ on B extending the A-action. Hence, for any $a\in\Gamma(A)$ and

$b_0, b_1, b_2 \in \Gamma(B)$, we have

$$
\begin{aligned}
0 =& \big(d^\nabla R^\nabla\big)(a, \widetilde{b}_0, \widetilde{b}_1) \\
=& \nabla_a\big(R^\nabla(\widetilde{b}_0, \widetilde{b}_1)\big) - \nabla_{\widetilde{b}_0}\big(R^\nabla(a, \widetilde{b}_1)\big) + \nabla_{\widetilde{b}_1}\big(R^\nabla(a, \widetilde{b}_0)\big) \\
& - R^\nabla([a, \widetilde{b}_0], \widetilde{b}_1) + R^\nabla([a, \widetilde{b}_1], \widetilde{b}_0) - R^\nabla([\widetilde{b}_0, \widetilde{b}_1], a) \\
=& \nabla_a\big(R^\nabla(\widetilde{b}_0, \widetilde{b}_1)\big) - \nabla_{\widetilde{b}_0}\big(R^\nabla(a, \widetilde{b}_1)\big) + \nabla_{\widetilde{b}_1}\big(R^\nabla(a, \widetilde{b}_0)\big) \\
& - R^\nabla(\widetilde{\nabla_a b_0}, \widetilde{b}_1) + R^\nabla(\Delta_{b_0} a, \widetilde{b}_1) + R^\nabla(\widetilde{\nabla_a b_1}, \widetilde{b}_0) - R^\nabla(\Delta_{b_1} a, \widetilde{b}_0) \\
& - R^\nabla\big(\alpha(b_0, b_1), a\big) - R^\nabla(\widetilde{\nabla_{\widetilde{b}_0} b_1}, a) + R^\nabla(\widetilde{\nabla_{\widetilde{b}_1} b_0}, a) + R^\nabla\big(\widetilde{\beta(b_0, b_1)}, a\big)
\end{aligned}
$$

and thus

$$
\begin{aligned}
0 =& \big(d^\nabla R^\nabla\big)(a, \widetilde{b}_0, \widetilde{b}_1) \cdot b_2 \\
=& \big(\partial^\nabla R_B^\nabla\big)(a; b_0, b_1) \cdot b_2 - \big(\partial^\nabla R_B^\nabla\big)(a; b_1, b_0) \cdot b_2 - R_B^\nabla\big(a, \beta(b_0, b_1)\big) \cdot b_2 \\
& + \nabla_a\big(R^\nabla(\widetilde{b}_0, \widetilde{b}_1)\big) \cdot b_2 - R^\nabla\big(\widetilde{\nabla_a b_0}, \widetilde{b}_1\big) \cdot b_2 - R^\nabla\big(\widetilde{b}_0, \widetilde{\nabla_a b_1}\big) \cdot b_2
\end{aligned}
$$

or, equivalently,

$$
0 = R_3(b_0, b_1, b_2) - R_3(b_1, b_0, b_2) - R_2\big(\beta(b_0, b_1), b_2\big) + \big(\partial^A \Omega\big)(b_0, b_1) \cdot b_2. \qquad \square
$$

Lemma 78. *For any $a \in \Gamma(A)$ and $b_0, b_1 \in \Gamma(B)$, we have*

$$
[\alpha(b_0, b_1), a] + \alpha(\nabla_a b_0, b_1) + \alpha(b_0, \nabla_a b_1) = \Delta_{b_0}\Delta_{b_1} a - \Delta_{b_1}\Delta_{b_0} a - \Delta_{\widetilde{q[\widetilde{b}_0, \widetilde{b}_1]}} a.
$$

Proof. We have

$$
\begin{aligned}
& p\big([\widetilde{b}_1, [\widetilde{b}_0, a]] + [\widetilde{b}_0, [a, \widetilde{b}_1]] + [a, [\widetilde{b}_1, \widetilde{b}_0]]\big) \\
=& p[\widetilde{b}_1, p[\widetilde{b}_0, a]] + p[\widetilde{b}_0, p[a, \widetilde{b}_1]] + p[a, p[\widetilde{b}_1, \widetilde{b}_0]] \\
& + p[\widetilde{b}_1, \widetilde{q[\widetilde{b}_0, a]}] + p[\widetilde{b}_0, \widetilde{q[a, \widetilde{b}_1]}] + p[a, \widetilde{q[\widetilde{b}_1, \widetilde{b}_0]}] \\
=& \Delta_{b_1}\Delta_{b_0} a - \Delta_{b_0}\Delta_{b_1} a + p[a, \alpha(b_1, b_0)] + p[\widetilde{\nabla_a b_0}, \widetilde{b}_1] + p[\widetilde{b}_0, \widetilde{\nabla_a b_1}] + p[\widetilde{q[\widetilde{b}_0, \widetilde{b}_1]}, a] \\
=& \Delta_{b_1}\Delta_{b_0} a - \Delta_{b_0}\Delta_{b_1} a + [\alpha(b_0, b_1), a] + \alpha(\nabla_a b_0, b_1) + \alpha(b_0, \nabla_a b_1) + \Delta_{\widetilde{q[\widetilde{b}_0, \widetilde{b}_1]}} a.
\end{aligned}
$$

The result follows from the Jacobi identity in L. $\square$

Lemma 79. *For any $n \geq 2$, $a \in \Gamma(A)$ and $b_0, b_1, \dots, b_n \in \Gamma(B)$, we have*

$$
\begin{aligned}
& R_{n+1}(a; b_0, b_1, b_2, \cdots, b_n) - R_{n+1}(a; b_1, b_0, b_2, \cdots, b_n) = \\
& \quad \Omega(b_0, b_1) \cdot R_{n-1}(a; b_2, \cdots, b_n) - \sum_{j=2}^{n} R_{n-1}\big(a; b_2, \cdots, \Omega(b_0, b_1) \cdot b_j, \cdots, b_n\big) \\
& \quad + \nabla_{\alpha(b_0, b_1)}\big(R_{n-1}(a; b_2, \cdots, b_n)\big) - \sum_{j=2}^{n} R_{n-1}\big(a; b_2, \cdots, \nabla_{\alpha(b_0, b_1)} b_j, \cdots, b_n\big) \\
& \quad - R_{n-1}\big([\alpha(b_0, b_1), a] + \alpha(\nabla_a b_0, b_1) + \alpha(b_0, \nabla_a b_1); b_2, \cdots, b_n\big) \\
& \quad + R_n\big(a; \beta(b_0, b_1), b_2, \cdots, b_n\big).
\end{aligned}
$$

Sketch of proof. Straightforward computation at the last step of which use is made of Lemma 78. $\square$

Proposition 80. *Let $L = A \bowtie B$ be a matched pair of Lie algebroids endowed with a flat torsion free B-connection on B. (These data determine a splitting of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and an L-connection on B extending the A-action such that the three associated bundle maps α, β, and Ω are trivial.) Then each $R_n : \otimes^n B \to \operatorname{Hom}(A, B)$ is totally symmetric in its n arguments.*

Proof. It follows from Proposition 55 and Lemma 77 that R_2 and R_3 are invariant under the permutation of their first two arguments. By Lemma 79, the same property holds for all higher R_n. Moreover, it is easy to see that, if R_n is symmetric in its n arguments, then R_{n+1} is symmetric in its last n arguments since $R_{n+1} = \partial^\nabla R_n$. The result follows by induction. □

Theorem 64, which says that $V[-1]$ is an L_∞-algebra when the assumptions of Proposition 80 are satisfied, is an immediate consequence of Proposition 80 and Theorem 57.

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